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Mixing-matrix renormalization revisited

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Abstract. We study the renormalization of normal mixing matrices, which include hermitian and unitary matrices as particular cases. We give a minimal, multiplicative parameterization of counterterms, and compute the renormalized Lagrangian to one-loop order in several simple models with N species of fermions, both in on-shell and $\overline{\rm MS}$ schemes. In the on-shell scheme the mass-degenerate case is considered separately.

1 Introduction

In theories with many particle species which mix non-trivially due to interactions, the mixing matrix generally requires renormalization like any other parameters in the Lagrangian. Such a renormalization has been considered in [1] within the context of the standard model (SM), and in extensions of the SM with Majorana neutrinos in [2].

In this paper we consider mixing-matrix renormalization in a more generic setting. Specifically, we study the renormalization of normal mixing matrices (i.e., matrices commuting with their adjoint), which include hermitian and unitary matrices as particular cases. We give a minimal, multiplicative parameterization of counterterms, and compute the renormalized Lagrangian to one-loop order in several simple models with N species of fermions, both in on-shell (OS) and $\overline{\rm MS}$ schemes. In the on-shell scheme the mass-degenerate case is considered separately. We work in dimensional regularization [3] throughout this paper.

Mixing-matrix renormalization in the SM, and in its extensions and low-energy effective theories, is usually closely related to other issues such as renormalization of theories with unstable particles, gauge invariance of the renormalization procedure, and CP violation. The latter is beyond the scope of this paper. We assume all particles to be stable. If that were not the case, one-loop self-energy parts should be replaced by their dispersive parts in OS renormalization conditions [4]. Gauge invariance of the renormalized CKM matrix in the SM has been considered in [5,6]. In our case, abelian gauge invariance plays a role in the discussion of unitary mixing matrices in Sect. 4.

In the next section we consider a model with N fermion flavors coupled to a scalar particle through a Yukawa interaction, specified by a hermitian coupling matrix. Renormalization of this model, and in particular the structure of its counterterms, is considered in detail and computed at the one-loop level. The results are generalized to normal matrices in Sect. 3. The particular case of unitary interaction matrices is treated in Sect. 4, where the relation between our parameterization of counterterms and the one commonly in use in the literature is discussed. In Sect. 5 we give some final remarks. We gather material relevant to all sections in four appendices. In Appendix A, in particular, we give a parameterization for mappings of normal matrices that we find useful in discussing renormalization of mass and mixing matrices.

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2 Hermitian mixing matrix

The simplest model of fermion mixing consists of N Dirac fields ψ_a, $a = 1, \ldots, N$, which we collect in a column field $\boldsymbol{\psi}$, interacting with a scalar field ϕ through a Yukawa coupling. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\phi_0(\Box + {m_{\phi}}_0^2)\phi_0 + \overline{\boldsymbol{\psi}}_0(\mathrm{i}\partial\!\!\!/ - \boldsymbol{M}_0)\boldsymbol{\psi}_0 + \overline{\boldsymbol{\psi}}_0\boldsymbol{H}_0\boldsymbol{\psi}_0\phi_0 - \frac{\xi_0}{3!}\phi_0^3 - \frac{\lambda_0}{4!}\phi_0^4, \tag{1}$$

the subindex 0 indicating bare fields and parameters. The Yukawa interaction term is specified by a matrix of couplings $\boldsymbol{H}$, whose elements are the expansion parameters in perturbation theory. $\boldsymbol{M}$ is the fermion mass matrix, which is assumed to be regular (i.e., no two of its eigenvalues are equal). The degenerate case will be considered below in Sect. 2.3. If $[\boldsymbol{M}, \boldsymbol{H}] \neq 0$, the interaction mixes flavors. Clearly, $\boldsymbol{M}$ and $\boldsymbol{H}$ must be hermitian. The interaction terms in ϕ^3 and ϕ^4 are needed for renormalizability. If fermions are massless and $\xi = 0$, $\mathcal{L}$ is invariant under the discrete symmetry $\boldsymbol{\psi} \to \gamma_5\boldsymbol{\psi}$, $\phi \to -\phi$ which forbids a ϕ^3 term. For massive fermions such a term cannot be avoided. Because of its simplicity and of its lack of symmetries preventing renormalization of its parameters, $\mathcal{L}$ is in some sense archetypal, so we consider its renormalization in detail below.

We never use the summation convention for flavor indices $a, b, c, \ldots$. Space-time indices are denoted by Greek letters, with summation over repeated indices always understood.

We write $\mathcal{L}$ in terms of renormalized fields and couplings by introducing renormalization constants. The field ϕ renormalizes multiplicatively: $\phi_0 = Z_\phi^{1/2}\phi = (1 + 1/2\delta Z_\phi)\phi$. In $\overline{\text{MS}}$ $Z_\phi = Z_\phi(\boldsymbol{H}, \lambda, d)$ [7], whereas in OS $Z_\phi = Z_\phi(\boldsymbol{H}, \lambda, d; m_\phi^2/\mu^2, \boldsymbol{M}/\mu, \xi/\mu)$. We do not expect multiplicative renormalization of λ, which already at one loop receives $\mathcal{O}(\boldsymbol{H}^4)$ divergent contributions from a box diagram, $\lambda_0 = \mu^\epsilon(\lambda + \delta\lambda)$. The dependence of $\delta\lambda$ on the parameters of the theory is analogous to that of Z_ϕ. In these expressions $d = 4 - \epsilon$ is the dimension of space-time and μ the mass scale of dimensional regularization.

The mass m_ϕ^2 will mix under renormalization with the other dimensionful parameters in $\mathcal{L}$. By dimensional analysis we have

$$m_{\phi_0}{}^2 = Z^{(1)}_{m_\phi} m_\phi^2 + \sum_{abcd} \delta Z^{(2)}_{abcd} M_{ab} M_{cd} + \delta Z^{(3)}_{m_\phi} \xi^2.$$

In OS renormalized masses are physical, so $M_{ab} = m_{a\,\text{ph}}\,\delta_{ab}$ and

$$m_{\phi_0}{}^2 = Z^{(1)}_{m_\phi} m_{\phi\,\text{ph}}{}^2 + \sum_{ab} \delta Z^{(2)}_{ab} m_{a\,\text{ph}} m_{b\,\text{ph}} + \delta Z^{(3)}_{m_\phi} \xi^2,$$

the renormalization constants $\delta Z^{(j)}_{m_\phi}$ depending on $\boldsymbol{H}$, λ, d and ratios of dimensionful parameters. In $\overline{\text{MS}}$ there is no dependence of $\delta Z^{(j)}_{m_\phi}$ on masses and ξ [7], but $\boldsymbol{M}$ is not diagonal. We can choose our flavor basis, however, so that at tree level $\boldsymbol{M}$ is diagonal. Off-diagonal elements in $\boldsymbol{M}$ are therefore of second order in $\boldsymbol{H}$, λ. At one loop we can then write

$$\delta m_\phi^2 = \delta Z^{(1)}_{m_\phi} m_\phi^2 + \sum_{ab} \delta Z^{(2)}_{ab} m_a m_b + \delta Z^{(3)}_{m_\phi} \xi^2 \qquad (2)$$

in both schemes, masses being physical in OS and renormalized ones in $\overline{\text{MS}}$. Similarly, for the cubic coupling we have

$$\xi_0 = \mu^{\epsilon/2} Z^{(1)}_\xi \xi + \mu^{\epsilon/2} \sum_{ab} \delta Z^{(2)}_{\xi,ab} M_{ab}, \qquad (3)$$

where we can set $M_{ab} = 0$ for $a \neq b$ at one loop. At that order, ξ mixes with $\boldsymbol{M}$ through a triangle diagram that does not depend on ξ or m_ϕ^2.

The renormalized fermion field can be related to the bare one as $\boldsymbol{\psi}_0 = \boldsymbol{A}\boldsymbol{\psi}$, with $\boldsymbol{A}$ a complex $N \times N$ matrix. It is more convenient to introduce the polar decomposition of $\boldsymbol{A}$ explicitly,

$$\boldsymbol{\psi}_0 = \boldsymbol{U}\boldsymbol{Z}^{1/2}\boldsymbol{\psi}, \quad \boldsymbol{U} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\delta U}}, \quad \boldsymbol{Z}^{1/2} = \boldsymbol{1} + \frac{1}{2}\boldsymbol{\delta Z}. \quad (4)$$

Both $\boldsymbol{\delta U}$, and $\boldsymbol{\delta Z}$ are hermitian. It is not difficult to show (see Appendix B), however, that we can always parameterize a unitary matrix $\boldsymbol{U}$ in a neighborhood of the identity as $\boldsymbol{U} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\delta U'}}\mathrm{e}^{-\mathrm{i}\boldsymbol{\delta\widetilde{U}}}$, with $\boldsymbol{\delta U'}$ a linear combination of diagonal generators and $\boldsymbol{\delta\widetilde{U}}$ a combination of the remaining ones. The effect of $\boldsymbol{\delta U'}$ is a flavor-dependent phase reparameterization of the fermion fields. Although $\mathcal{L}$ is not invariant under such transformations, physical quantities (such as S-matrix elements) remain unaffected by them. We choose to set $\boldsymbol{\delta U'} = 0$. In fact, as we shall see below, in OS renormalization conditions do not determine the diagonal elements of $\boldsymbol{\delta U}$. (Thus, in (4) we have $\boldsymbol{\delta Z} \in \boldsymbol{u}(N)$, whereas $\boldsymbol{\delta U} \in \boldsymbol{u}(N)/\boldsymbol{a}$, with $\boldsymbol{u}(N)$ the Lie algebra of $N \times N$ hermitian matrices, and $\boldsymbol{a}$ its Cartan subalgebra of diagonal generators.)

The renormalization matrices $\boldsymbol{U}$ and $\boldsymbol{Z}^{1/2}$ have different roles in the theory. If, for instance, we take ψ_a to be scalar fields and set $\lambda = 0$ in $\mathcal{L}$, $\boldsymbol{Z}^{1/2}$ turns out to be finite, as we would expect of wave-function renormalization in a superrenormalizable theory [8]. On the other hand, in OS, mass renormalization contributions make $\boldsymbol{U}$ divergent. If, furthermore, the scalar fields are real, $\boldsymbol{Z}$ must be real symmetric and $\boldsymbol{U}$ orthogonal. In this case $\mathrm{i}\boldsymbol{\delta U}$ is antisymmetric, with diagonal elements vanishing by definition, which is consistent with the fact that no phase redefinitions are possible for real fields.

The fermion mass matrix mixes under renormalization with ξ. Following the parameterization given in Appendix A, we write

$$\begin{aligned} &\boldsymbol{M}_0 = \boldsymbol{U}_m^\dagger \left(\boldsymbol{M} + \overline{\boldsymbol{\delta}} M\right) \boldsymbol{U}_m + \boldsymbol{\delta Z}_m^{(\xi)} \xi, \quad \left[\overline{\boldsymbol{\delta}} M, \boldsymbol{M}\right] = \boldsymbol{0}, \\ &\boldsymbol{U}_m = \mathrm{e}^{-\mathrm{i}\boldsymbol{\delta U}_m}, \end{aligned} \qquad (5)$$

with $\overline{\boldsymbol{\delta}} M, \boldsymbol{\delta U}_m, \boldsymbol{\delta Z}_m^{(\xi)}$ hermitian. Here we have conventionally chosen an additive parameterization for the first term in $\boldsymbol{M}_0$. Since we treat ξ as a perturbative parameter, we collect all dependence on it in the second term. At one loop $\boldsymbol{\delta Z}_m^{(\xi)} = 0$, so we can write $\boldsymbol{M}_0 = \boldsymbol{M} + \boldsymbol{\delta M}$ with $\boldsymbol{\delta M} = \overline{\boldsymbol{\delta}} \boldsymbol{M} + \mathrm{i}\,[\boldsymbol{\delta U}_m, \boldsymbol{M}]$. $\boldsymbol{\delta U}_m$ is only determined up to addition of a matrix commuting with $\boldsymbol{M}$.

Finally, the matrix of Yukawa couplings $\boldsymbol{H}$ is multiplicatively renormalized. The most general multiplicative transformation that can be applied to $\boldsymbol{H}$ preserving hermiticity is of the form (see Appendix A)

$$\begin{aligned} &\boldsymbol{H}_0 = \mu^{\epsilon/2} \boldsymbol{W}^\dagger \boldsymbol{Z}_H \boldsymbol{H} \boldsymbol{W}, \quad [\boldsymbol{Z}_H, \boldsymbol{H}] = 0, \\ &\boldsymbol{W} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\delta W}}, \end{aligned} \qquad (6)$$

with $\boldsymbol{Z}_H$, $\boldsymbol{\delta W}$ hermitian. In $\overline{\text{MS}}$ $\boldsymbol{Z}_H$ and $\boldsymbol{W}$ can depend only on $\boldsymbol{H}$, λ and d, so that $\boldsymbol{W}$ commutes with both $\boldsymbol{Z}_H$ and $\boldsymbol{H}$, and drops from (6). This would not be the case, though, if there were another dimensionless matrix in $\mathcal{L}$ on which $\boldsymbol{W}$ could depend such as, e.g., another mixing matrix for an additional family of fermions.

Substituting these definitions into (1) we see that, out of the three unitary matrices we have introduced, only two combinations enter $\mathcal{L}$, namely, $\boldsymbol{U}_m\boldsymbol{U}$ and $\boldsymbol{W}\boldsymbol{U}$. We can therefore always choose $\boldsymbol{U} = \boldsymbol{1}$. In OS, however, we parameterize the theory so that $\boldsymbol{M}_0$ and $\boldsymbol{M}$ are simultaneously diagonal by setting $\boldsymbol{U}_m = \boldsymbol{1}$.

The Lagrangian can then be written in terms of renormalized fields and counterterms as

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m_\phi^2)\phi - \frac{1}{2}\delta Z_\phi \phi \Box \phi - \frac{1}{2}\Delta m_\phi^2 \phi^2$$

$$+ \overline{\psi}(\mathrm{i}\partial\!\!\!/ - \boldsymbol{M})\psi + \overline{\psi}\boldsymbol{\delta Z} i\partial\!\!\!/\psi - \overline{\psi}\boldsymbol{\Delta M}\psi + \mu^{\epsilon/2}\overline{\psi}\left(\boldsymbol{H} + \boldsymbol{\Delta H}\right)\psi\phi - \frac{1}{3!}\mu^{\epsilon/2}(\xi + \Delta\xi)\phi^3 - \frac{1}{4!}\mu^{\epsilon}(\lambda + \Delta\lambda)\phi^4, \tag{7}$$

with the one-loop counterterms defined as

$$\Delta m_\phi^2 = \delta Z_\phi m_\phi^2 + \delta m_\phi^2, \quad \Delta\lambda = \delta\lambda + 2\lambda\delta Z_\phi, \tag{8a}$$

$$\Delta\xi = \delta\xi + \frac{3}{2}\xi\delta Z_\phi, \quad \delta\xi = \delta Z_\xi^{(1)}\xi + \sum_a \delta Z_{\xi,a}^{(2)} m_a, \tag{8b}$$

$$\boldsymbol{\Delta M} = \frac{1}{2}\{\boldsymbol{M}, \boldsymbol{\delta Z}\} + \overline{\delta} M + \mathrm{i}\left[\boldsymbol{\delta U} + \boldsymbol{\delta U}_m, \boldsymbol{M}\right], \tag{8c}$$

$$\boldsymbol{\Delta H} = \boldsymbol{\delta Z}_H \boldsymbol{H} + \mathrm{i}\left[\boldsymbol{\delta U} + \boldsymbol{\delta W}, \boldsymbol{H}\right] + \frac{1}{2}\{\boldsymbol{\delta Z}, \boldsymbol{H}\} + \frac{1}{2}\delta Z_\phi \boldsymbol{H}. \tag{8d}$$

In order to compute the renormalization parts of the theory we have first to fix a renormalization scheme.

2.1 Overspecified counterterm parameterization

In (5) and (6) we have given the relation between bare and renormalized hermitian matrices using the parameterization in Appendix A, which is applicable to generic normal matrices. In the particular case of hermitian matrices a more obvious multiplicative parameterization is the congruence transformation $\boldsymbol{H}_0 = \boldsymbol{A}^\dagger \boldsymbol{H} \boldsymbol{A}$, with $\boldsymbol{A}$ non-singular and dependent on $\boldsymbol{H}$. Such a parameterization must satisfy constraint relations, since it is overspecified in the following sense.

Let $\boldsymbol{f} : \boldsymbol{u}(N) \longrightarrow \boldsymbol{u}(N)$ be a map of hermitian matrices, given by $\boldsymbol{f}(\boldsymbol{H}) = \boldsymbol{A}^\dagger \boldsymbol{H} \boldsymbol{A}$, with $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{H})$ non-singular. Then we can find a non-singular $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H})$ such that $\boldsymbol{f}(\boldsymbol{H}) = \boldsymbol{B}^\dagger \boldsymbol{H} \boldsymbol{B}$, and

$$\left[\boldsymbol{B}\boldsymbol{B}^\dagger, \boldsymbol{H}\right] = 0, \quad \left[\boldsymbol{B}^\dagger \boldsymbol{B}, \boldsymbol{f}(\boldsymbol{H})\right] = 0. \tag{9}$$

To see this, we notice that according to Appendix A there must be a hermitian matrix $\boldsymbol{Z}_H$ commuting with $\boldsymbol{H}$ and a unitary matrix $\boldsymbol{U}$ such that $\boldsymbol{f}(\boldsymbol{H}) = \boldsymbol{U}^\dagger \boldsymbol{Z}_H \boldsymbol{H} \boldsymbol{U}$. Defining $\boldsymbol{B} = \boldsymbol{Z}_H^{1/2}\boldsymbol{U}$, we obtain the result. The parameterization given in Appendix A is minimal in the sense that (9) are satisfied identically.

2.2 On-shell scheme

In OS we set $\boldsymbol{U}_m = \boldsymbol{1}$. If $\boldsymbol{M}$ is regular and diagonal, and $[\boldsymbol{M}, \overline{\boldsymbol{\delta}}\boldsymbol{M}] = 0$, then $\overline{\delta}\boldsymbol{M}$ must be diagonal, as well as $\boldsymbol{M}_0$.

The renormalization conditions we impose on the scalar field self-energy $\Pi_\phi(p^2)$ are $\Pi_\phi(m_\phi^2) = 0 = \Pi'_\phi(m_\phi^2)$, where the prime stands for $\partial/\partial p^2$. Defining

$$\Omega_\phi(p^2) = -\frac{1}{8\pi^2}\sum_{a,b} H_{ab}H_{ba}\left(m_a^2 a_0(m_a^2) + m_b^2 a_0(m_b^2) + \left((m_a + m_b)^2 - p^2\right) b_0(p^2, m_a^2, m_b^2)\right), \tag{10}$$

which is the $\mathcal{O}(\epsilon^0)$ part of the unrenormalized ϕ self-energy, we obtain

$$\Pi_\phi(p^2) = \Omega_\phi(p^2) - \Omega_\phi(m_\phi^2) - (p^2 - m_\phi^2)\Omega'_\phi(m_\phi^2), \tag{11a}$$

$$\delta Z_\phi = -\frac{1}{4\pi^2\epsilon}\mathrm{Tr}(\boldsymbol{H}^2) + \Omega'_\phi(m_\phi^2), \tag{11b}$$

$$\delta m_\phi^2 = \frac{\xi^2}{16\pi^2\epsilon} - \frac{1}{2\pi^2\epsilon}\left(2\mathrm{Tr}(\boldsymbol{H}^2\boldsymbol{M}^2) + \mathrm{Tr}\left((\boldsymbol{H}\boldsymbol{M})^2\right)\right) - \Omega_\phi(m_\phi^2) + m_\phi^2\Omega'_\phi(m_\phi^2). \tag{11c}$$

In (10), a_0 and b_0 refer to finite parts of loop integrals, defined in Appendix D. (All dependence on the renormalization scale μ is through these integrals and, as is easy to check, $\Pi_\phi(p^2)$ does not depend on μ as it should in OS.) All masses in this section are physical (pole) masses.

For the fermion two-point function, which is a matrix in flavor space, we write

$$\boldsymbol{\Gamma} = p\!\!\!/\boldsymbol{1} - \boldsymbol{M} - \boldsymbol{\Pi}(p), \quad \boldsymbol{\Pi}(p) = p\!\!\!/\boldsymbol{\Sigma}^V(p^2) + \boldsymbol{\Sigma}^S(p^2), \tag{12a}$$

where the form factors are given by

$$\boldsymbol{\Sigma^V}(p^2) = -\frac{1}{16\pi^2\epsilon}\boldsymbol{H}^2 - \boldsymbol{\delta Z} + \boldsymbol{\Omega}^V(p^2),$$

$$\boldsymbol{\Sigma^S}(p^2) = -\frac{1}{8\pi^2\epsilon}\boldsymbol{H}\boldsymbol{M}\boldsymbol{H} + \boldsymbol{\Delta M} + \boldsymbol{\Omega}^S(p^2), \tag{12b}$$

with

$$\Omega^V_{ab}(p^2) = \frac{1}{16\pi^2}\sum_c H_{ac}H_{cb}b_-(p^2, m_\phi^2, m_c^2),$$

$$\Omega^S_{ab}(p^2) = \frac{1}{16\pi^2}\sum_c H_{ac}H_{cb}m_c b_0(p^2, m_\phi^2, m_c^2). \tag{12c}$$

Renormalization conditions are expressed in terms of $\boldsymbol{\Sigma}^{V,S}(p^2)$ by [4]

$$m_a\Sigma^V_{aa}(m_a^2) + \Sigma^S_{aa}(m_a^2) = 0, \quad \Sigma^V_{aa}(m_a^2) + 2m_a^2\Sigma^{V\prime}_{aa}(m_a^2) + 2m_a\Sigma^{S\prime}_{aa}(m_a^2) = 0, \tag{13a}$$

$$m_b\Sigma^V_{ab}(m_b^2) + \Sigma^S_{ab}(m_b^2) = 0, \quad m_a\Sigma^V_{ab}(m_a^2) + \Sigma^S_{ab}(m_a^2) = 0, \tag{13b}$$

where in the two last equations $a \neq b$. With these conditions, we obtain for the diagonal two-point functions,

$$\Sigma^V_{aa}(p^2) = \Omega^V_{aa}(p^2) - \Omega^V_{aa}(m_a^2) - 2m_a^2\Omega^{V\prime}_{aa}(m_a^2) - 2m_a\Omega^{S\prime}_{aa}(m_a^2) \tag{14a}$$

$$\Sigma^S_{aa}(p^2) = \Omega^S_{aa}(p^2) - \Omega^S_{aa}(m_a^2) + 2m_a^3\Omega^{V\prime}_{aa}(m_a^2) + 2m_a^2\Omega^{S\prime}_{aa}(m_a^2), \tag{14b}$$

and for the off-diagonal ones ($a \neq b$)

$$\Sigma^V_{ab}(p^2) = \Omega^V_{ab}(p^2) - \frac{m_a\Omega^V_{ab}(m_a^2) - m_b\Omega^V_{ab}(m_b^2)}{m_a - m_b} - \frac{\Omega^S_{ab}(m_a^2) - \Omega^S_{ab}(m_b^2)}{m_a - m_b}, \tag{14c}$$

$$\Sigma^S_{ab}(p^2) = \Omega^S_{ab}(p^2) + \frac{m_a m_b}{m_a - m_b}\left(\Omega^V_{ab}(m_a^2) - \Omega^V_{ab}(m_b^2)\right) + \frac{m_b\Omega^S_{ab}(m_a^2) - m_a\Omega^S_{ab}(m_b^2)}{m_a - m_b}. \quad (14d)$$

In both cases, diagonal and off-diagonal, renormalization constants can be compactly expressed as

$$\boldsymbol{\delta Z} = -\frac{1}{16\pi^2\epsilon}\boldsymbol{H}^2 + \boldsymbol{\Omega}^V(p^2) - \boldsymbol{\Sigma}^V(p^2),$$
$$\boldsymbol{\Delta M} = \frac{1}{8\pi^2\epsilon}\boldsymbol{H}\boldsymbol{M}\boldsymbol{H} - \boldsymbol{\Omega}^S(p^2) + \boldsymbol{\Sigma}^S(p^2). \quad (15)$$

It is immediate to check, by substituting the expressions for $\boldsymbol{\Sigma}^{V,S}$ into these equations, that $\boldsymbol{\delta Z}$ and $\boldsymbol{\Delta M}$ are hermitian. With this, the fermion propagator is renormalized to one loop. We can, however, compute the mass counterterm $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ and $\boldsymbol{\delta U}$ from the expressions (15) for $\boldsymbol{\Delta M}$ and $\boldsymbol{\delta Z}$. From (8c) we have

$$\overline{\boldsymbol{\delta}}\boldsymbol{M} + \mathrm{i}[\boldsymbol{\delta U}, \boldsymbol{M}] = \boldsymbol{\Delta M} - \frac{1}{2}\{\boldsymbol{\delta Z}, \boldsymbol{M}\}. \quad (16)$$

This equation can always be solved for $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ and $\boldsymbol{\delta U}$, as discussed in Appendix C. In this case, in which $\boldsymbol{M}$ is diagonal and regular, however, (16) is trivial to solve. $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ is given by the diagonal elements of the r.h.s. of (16) and $\boldsymbol{\delta U}$ by the off-diagonal ones,

$$\overline{\delta}M_{ab} = (\Delta M_{aa} - m_a\delta Z_{aa})\delta_{ab}, \quad (17a)$$
$$\delta U_{ab} = \frac{\mathrm{i}}{m_a - m_b}\left(\Delta M_{ab} - \frac{m_a + m_b}{2}\delta Z_{ab}\right), \quad a \neq b. \quad (17b)$$

The diagonal elements δU_{aa} are not determined by renormalization conditions. As mentioned above, they only change the phases of fermion fields, and can be chosen to vanish.

We consider next the renormalization of the Yukawa coupling, given by the 1-PI three-point Green's function $\boldsymbol{\Gamma}(p_1, p_2)$. (Here p_1 is the momentum of the incoming scalar and $p_{2,3}$ are the momenta of the outgoing fermions.) At tree level $\boldsymbol{\Gamma} = \boldsymbol{H}$ is a Lorentz scalar. Expanding the one-loop $\boldsymbol{\Gamma}$ in the usual γ-matrix basis, it is clear that only the scalar form factor can receive divergent contributions, since the counterterm for $\boldsymbol{\Gamma}$ in $\mathcal{L}$ is a scalar, the other form factors being finite. This is easily seen also from the explicit form of the corresponding loop integrals, since by power counting only the terms containing two powers of loop momentum in the numerator are divergent, and they contribute to the scalar form factor only.

We will then focus on $\boldsymbol{F}(p_1, p_2) \equiv 1/4\mathrm{Tr}(\boldsymbol{\Gamma}(p_1, p_2))$. For concreteness, we assume that the physical value of the coupling $\boldsymbol{H}$ is fixed by this form factor. $\boldsymbol{F}$ is a scalar function of momenta, depending on p_1, p_2 only through p_1^2, p_2^2, $(p_1 - p_2)^2$. These combinations are fixed if the external momenta are required to be on their mass shell. Calling $F^{(\mathrm{os})}_{ab}$ the value of $F_{ab}(p_1, p_2)$ at $p_1^2 = m_\phi^2$, $p_2^2 = m_a^2$, $(p_1 - p_2)^2 = m_b^2$, we impose the renormalization condition

$$\boldsymbol{F}^{(\mathrm{os})} = \boldsymbol{H}. \quad (18)$$

(We notice, furthermore, that the hermiticity of the l.h.s. is guaranteed by CPT invariance, which also requires $\boldsymbol{H}$ to be hermitian.) At one loop $\boldsymbol{F}(p_1, p_2)$ is given by

$$F_{ab}(p_1, p_2) = \mu^{\epsilon/2}(H_{ab} + \Delta H_{ab}) - \mu^{\epsilon/2}\frac{1}{8\pi^2\epsilon}\left(\boldsymbol{H}^3\right)_{ab} + \mu^{\epsilon/2}\sum_{c,d} H_{ac}H_{cd}H_{db}h_1(p_1, p_2; m_\phi^2, m_c^2, m_d^2) - \mu^{\epsilon/2}\xi\sum_c H_{ac}H_{cb}h_2(p_1, p_2; m_\phi^2, m_c^2), \quad (19)$$

where the loop integrals $h_{1,2}$ are defined in Appendix D. The renormalization condition (18) then leads to

$$\Delta H_{ab} = \frac{1}{8\pi^2\epsilon}\left(\boldsymbol{H}^3\right)_{ab} - \sum_{cd} H_{ac}H_{cd}H_{db}h_1^{(\mathrm{os})}(m_\phi^2, m_c^2, m_d^2) + \xi\sum_c H_{ac}H_{cb}h_2^{(\mathrm{os})}(m_\phi^2, m_c^2), \quad (20)$$

where $h_{1,2}^{(\mathrm{os})}$ refer to the on-shell values of those integrals. This counterterm is enough to renormalize the scalar form factor at one loop. We can, in principle, determine the counterterms to the coupling matrix $\boldsymbol{H}$ as defined in (8d) through the equation

$$\boldsymbol{\delta Z}_H\boldsymbol{H} + \mathrm{i}\left[\boldsymbol{\delta W}, \boldsymbol{H}\right] = \boldsymbol{\Delta H} - \mathrm{i}\left[\boldsymbol{\delta U}, \boldsymbol{H}\right] - \frac{1}{2}\left\{\boldsymbol{\delta Z}, \boldsymbol{H}\right\} - \frac{1}{2}\delta Z_\phi\boldsymbol{H}, \quad (21)$$

where all the quantities on the r.h.s. are already known. Equation (21) always has a solution, since by definition $[\boldsymbol{\delta Z}_H, \boldsymbol{H}] = 0$, as shown in Appendix C. Unlike (16), in this case it is more difficult to find the solution algebraically in closed form, only some of the contributions to each counterterm being obvious,

$$\boldsymbol{\delta Z}_H = \frac{1}{8\pi^2\epsilon}\left(\boldsymbol{H}^2\right) - \frac{1}{2}\delta Z_\phi + \cdots,$$
$$\left[\boldsymbol{\delta W}, \boldsymbol{H}\right] = -\left[\boldsymbol{\delta U}, \boldsymbol{H}\right] + \cdots \quad (22)$$

Equation (21) can be solved by projecting it over an appropriate basis for the algebra $\boldsymbol{u}(N)$ (see Appendix C), which can be done numerically. It should be clear, however, that counterterms are completely fixed by renormalization conditions. Once those are established, no other choices are involved. We notice that diagonal elements of $\boldsymbol{\delta U}$ contribute to $\boldsymbol{\delta W}$, as expected, since a change of phase of fermion fields leads to a corresponding change in $\boldsymbol{H}$.

2.3 The mass-degenerate case in on-shell scheme

As discussed in Sect. 2.2 in OS we set $\boldsymbol{U}_m = \boldsymbol{1}$, so that $\boldsymbol{M}_0 = \boldsymbol{M} + \overline{\boldsymbol{\delta}}\boldsymbol{M}$, with $[\boldsymbol{M}, \overline{\boldsymbol{\delta}}\boldsymbol{M}] = 0$. $\boldsymbol{M}_0$ and $\boldsymbol{M}$ can then be simultaneously diagonalized. If $\boldsymbol{M}$ is regular, choosing a basis in which it is diagonal immediately implies that $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ is diagonal, and then so is $\boldsymbol{M}_0$. If $\boldsymbol{M}$ is degenerate, though, $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ need not be diagonal even if $\boldsymbol{M}$ is. We notice

also that $\boldsymbol{H}$ can be diagonalized within each eigenspace of $\boldsymbol{M}$ by means of a unitary transformation of $\mathcal{O}(\boldsymbol{H}^0)$. We assume that such a transformation has been performed and then, in general, $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ will not be diagonal within the mass eigenspaces.

For notational simplicity we assume that there is one degenerate mass eigenvalue, say m_1, with multiplicity $1 < r < N$ ($r = N$ being the trivial case $\boldsymbol{M} \propto \boldsymbol{1}$), all other masses, $m_{r+1}, \ldots, m_N$, being non-degenerate. The generalization to the case of several degenerate eigenvalues presents no difficulties.

The renormalized fermion two-point Green's functions, as given by (14) and (15) are unchanged in the mass-degenerate case if $a = b$ or if $a \neq b$ but $\max(a, b) > r$. If $a \neq b$ and both $a, b \leq r$, those results need modification.

The renormalization conditions (13b) reduce to the single equation ($a \neq b$, $a, b \leq r$)

$$m_1 \Sigma^V_{ab}(m_1^2) + \Sigma^S_{ab}(m_1^2) = 0, \tag{23}$$

leading to

$$\begin{aligned} \Delta M_{ab} - m_1 \delta Z_{ab} = {} & \frac{m_1}{16\pi^2\epsilon} \left(\boldsymbol{H}^2\right)_{ab} + \frac{1}{8\pi^2\epsilon}(\boldsymbol{HMH})_{ab} \\ & - m_1 \Omega^V_{ab}(m_1^2) - \Omega^S_{ab}(m_1^2). \end{aligned} \tag{24}$$

From definition (8c) we see that in this case, $\Delta M_{ab} - m_1 \delta Z_{ab} = \overline{\delta} M_{ab}$. Thus, the renormalization condition (23) fixes the off-diagonal counterterm $\overline{\delta} M_{ab}$ to the value given in (24), and that is all that is required to renormalize the two-point function within the eigenspace of m_1. $\overline{\delta} M_{ab}$, as given by (17a) when $a = b$ or $a \neq b$, $\max(a, b) > r$, and by (24), when $a \neq b$ and $a, b \leq r$, obviously commutes with $\boldsymbol{M}$, since it is non-diagonal only within eigenspaces of $\boldsymbol{M}$. Furthermore, we set $\delta U_{ab} = 0$, for $a, b \leq r$, instead of (17b). Notice that when $r = N$, $\boldsymbol{H}$ is diagonal and therefore so are $\boldsymbol{\Omega}^{V,S}$ and $\overline{\boldsymbol{\delta}}\boldsymbol{M}$.

If, however, we want to make each form factor $\Sigma^{V,S}_{ab}$ finite separately, we may impose additional renormalization conditions. These are quite arbitrary, as long as they are consistent with (23) (or (24)). We can, for instance, take the limit of degenerate masses in (14) and (15) to get, for $a \neq b$, $a, b \leq r$,

$$\begin{aligned} \delta Z_{ab} = {} & -\frac{1}{16\pi^2\epsilon}(\boldsymbol{H}^2)_{ab} + \Omega^V_{ab}(m_1^2) + 2m_1^2 \Omega^{V\prime}_{ab}(m_1^2) \\ & + 2m_1 \Omega^{S\prime}_{ab}(m_1^2), \end{aligned} \tag{25a}$$

$$\begin{aligned} \Delta M_{ab} = {} & \frac{1}{8\pi^2\epsilon}(\boldsymbol{HMH})_{ab} - \Omega^S_{ab}(m_1^2) + 2m_1^3 \Omega^{V\prime}_{ab}(m_1^2) \\ & + 2m_1^2 \Omega^{S\prime}_{ab}(m_1^2). \end{aligned} \tag{25b}$$

Requiring instead $\Sigma^{V,S}_{ab}(m_1^2) = 0$ leads to

$$\begin{aligned} \delta Z_{ab} &= -\frac{1}{16\pi^2\epsilon}(\boldsymbol{H}^2)_{ab} + \Omega^V_{ab}(m_1^2), \\ \Delta M_{ab} &= \frac{1}{8\pi^2\epsilon}(\boldsymbol{HMH})_{ab} - \Omega^S_{ab}(m_1^2). \end{aligned} \tag{26}$$

Both (25) and (26) are consistent with (23).

2.4 $\overline{\text{MS}}$ scheme

In $\overline{\text{MS}}$ we set $\boldsymbol{U} = \boldsymbol{1}$. We choose a flavor basis for fermion fields so that at tree level the renormalized mass matrix $\boldsymbol{M}$ is diagonal, $\boldsymbol{M} = \boldsymbol{M}_{\text{phys}}$. Off-diagonal elements of $\boldsymbol{M}$ are then $\mathcal{O}(\boldsymbol{H}^2)$. We write $\boldsymbol{M} = \boldsymbol{M'} + \widehat{\boldsymbol{M}}$, with $\boldsymbol{M'} = \text{diag}(m_1, \ldots, m_N)$ containing the renormalized masses and $\widehat{\boldsymbol{M}}$ the off-diagonal elements, and treat $\widehat{\boldsymbol{M}}$ as an interaction term. We write the fermion Lagrangian as

$$\begin{aligned} \mathcal{L}_\psi &= \overline{\boldsymbol{\psi}}(\mathrm{i}\partial\!\!\!/ - \boldsymbol{M'})\boldsymbol{\psi} - \overline{\boldsymbol{\psi}}\widehat{\boldsymbol{M}}\boldsymbol{\psi} + \overline{\boldsymbol{\psi}}(\boldsymbol{\delta Z}\mathrm{i}\partial\!\!\!/ - \boldsymbol{\Delta M})\boldsymbol{\psi}, \\ \boldsymbol{\Delta M} &= \overline{\boldsymbol{\delta}}\boldsymbol{M} + \mathrm{i}[\boldsymbol{\delta U}_m, \boldsymbol{M}] + \frac{1}{2}\{\boldsymbol{M}, \boldsymbol{\delta Z}\}, \end{aligned}$$

instead of the second line of (7). The tree-level fermion propagator is thus flavor diagonal.

We notice, parenthetically, that $\boldsymbol{M}$ can be written as $\boldsymbol{M} = \exp(\mathrm{i}\boldsymbol{E})\boldsymbol{M'}\exp(-\mathrm{i}\boldsymbol{E})$, with $\boldsymbol{M'}$ the diagonal matrix of eigenvalues and $\exp(-\mathrm{i}\boldsymbol{E})$ the unitary matrix of eigenvectors of $\boldsymbol{M}$. Our choice of tree-level flavor basis implies $\boldsymbol{E} = \mathcal{O}(\boldsymbol{H}^2)$. At one-loop level, then, $\boldsymbol{M'}$ is given by the diagonal entries of $\boldsymbol{M}$ and $\widehat{\boldsymbol{M}} = \mathrm{i}[\boldsymbol{E}, \boldsymbol{M'}]$ contains the off-diagonal ones.

Counterterms for the scalar two-point function, as defined by (7) and (8a), are given by the ϵ-pole terms of (11b) and (11c). The ϕ self-energy is then $\Pi_\phi(p^2) = \Omega_\phi(p^2)$, with Ω_ϕ defined in (10). Requiring that the two-point function have a zero at $p^2 = m^2_{\phi\text{ph}}$ leads to the relation $m^2_\phi = m^2_{\phi\text{ph}} - \Omega_\phi(m^2_{\phi\text{ph}})$ at one loop.

The fermion two-point function can also be read off the corresponding OS results. Starting from (12), we set $\boldsymbol{\delta Z}$ and $\boldsymbol{\Delta M}$ to be the ϵ poles of (15), to obtain $\boldsymbol{\Sigma}^{V,S}(p^2) = \boldsymbol{\Omega}^{V,S}(p^2)$. The counterterms $\overline{\boldsymbol{\delta}}\boldsymbol{M}$ and $\boldsymbol{\delta U}_m$ can then be obtained from $\boldsymbol{\delta Z}$ and $\boldsymbol{\delta M}$ in the same way as in OS (see (16)). In this case, unlike in OS, they are not needed to renormalize the Yukawa coupling. We define

$$\begin{aligned} \boldsymbol{\delta M} &= \boldsymbol{\Delta M} - \frac{1}{2}\{\boldsymbol{M}, \boldsymbol{\delta Z}\} \\ &= \frac{1}{8\pi^2\epsilon}\left(\boldsymbol{HMH} + \frac{1}{4}\left\{\boldsymbol{M}, \boldsymbol{H}^2\right\}\right), \end{aligned} \tag{27}$$

which we will use in the one-loop renormalization group (RG) equation for $\boldsymbol{M}$ below.

We require $\Gamma_{aa}(p)u_a(p)|_{p^2 = m^2_{\text{aph}}} = 0$ in order to obtain renormalized masses in terms of physical ones, leading to the relation

$$m_a = m_{\text{aph}} - \left(m_{\text{aph}}\Omega^V_{aa}(m^2_{\text{aph}}) + \Omega^S_{aa}(m^2_{\text{aph}})\right). \tag{28}$$

Expressing the off-diagonal elements $\widehat{M}_{ab}$, $a \neq b$, in terms of physical masses and coupling constants involves a choice of parameterization of the theory. In principle, any value of $\widehat{\boldsymbol{M}}$ consistent with the renormalization group equations (given below) is admissible. Different choices of $\widehat{\boldsymbol{M}}$ at one-loop level will result in different parameterizations of the renormalized masses m_a (the eigenvalues of $\boldsymbol{M}$), in terms of the physical ones m_{aph} at two loops. We could give

$\widehat{\boldsymbol{M}}$ a definite value at a mass scale $\overline{\mu}_0$, and evolve it with the RG equations to the desired scale $\overline{\mu}$. For concreteness, we quote the expression for $\boldsymbol{M}$ obtained by matching the theory in $\overline{\rm MS}$ to the OS results,

$$M_{ab} = (\boldsymbol{M}_{\rm ph})_{ab} - \frac{1}{2}\left(m_{a\rm ph}\Omega^V_{ab}(m^2_{a\rm ph}) + \Omega^S_{ab}(m^2_{a\rm ph}) + m_{b\rm ph}\Omega^V_{ab}(m^2_{b\rm ph}) + \Omega^S_{ab}(m^2_{b\rm ph})\right), \quad (29)$$

where $\boldsymbol{M}_{\rm ph} = {\rm diag}(m_{1\rm ph},\dots,m_{N\rm ph})$. When $a = b$ this equation reduces to (28).

It is convenient to discuss at this point the effect of a finite, unitary renormalization of fermion fields, with $\boldsymbol{\delta U} = \mathcal{O}(\boldsymbol{H}^2)$ and d-independent. (We exclude from consideration unitary transformations with $\boldsymbol{\delta U} = \mathcal{O}(\boldsymbol{H}^0)$, which correspond to transformations of the classical fields.) The effect of such a transformation on the basic Lagrangian is to change $\boldsymbol{M} \to \boldsymbol{U}^\dagger\boldsymbol{M}\boldsymbol{U}$ and $\boldsymbol{H} \to \boldsymbol{U}^\dagger\boldsymbol{H}\boldsymbol{U}$. It is clear that $\boldsymbol{M}$ remains diagonal at $\mathcal{O}(\boldsymbol{H}^0)$, so our choice of a tree-level flavor basis is not altered. Also, $\boldsymbol{M}'$ remains unchanged through $\mathcal{O}(\boldsymbol{H}^2)$, so (28) still holds. Changes in $\boldsymbol{M}'$ start at $\mathcal{O}(\boldsymbol{H}^4)$. On the other hand, $\widehat{M}$ changes by a term $i[\boldsymbol{\delta U}, \boldsymbol{M}']$ at $\mathcal{O}(\boldsymbol{H}^2)$. Counterterms still are of the $\overline{\rm MS}$ form after the transformation, which is multiplicative and independent of ϵ. (If, however, $\boldsymbol{\delta U}$ depends on masses, counterterms acquire mass dependence.) We see, then, that this $U(N)$ freedom to perform finite unitary renormalizations of $\boldsymbol{\psi}$ is a source of arbitrariness in $\widehat{\boldsymbol{M}}$. By choosing a definite value for $\boldsymbol{M}$, as in (29), we are reducing the ambiguity in the choice of one-loop flavor basis for $\boldsymbol{\psi}$ to the subgroup of $U(N)$ that leaves $\boldsymbol{M}$ invariant. If $\boldsymbol{M}$ is regular, this is the $(U(1))^N$ abelian subgroup of flavor-dependent phase transformations.

The relation between bare and renormalized Yukawa couplings is given by (6). As discussed in relation to that equation, the unitary renormalization matrix $\boldsymbol{W}$ is trivial in $\overline{\rm MS}$ in this model. From the value of $\boldsymbol{\Delta H}$ in OS, (20), we get

$$\boldsymbol{\Delta H} = \frac{1}{8\pi^2\epsilon}\boldsymbol{H}^3.$$

Therefore, from definition (8d) we obtain,

$$\boldsymbol{\delta Z}_H = \frac{1}{16\pi^2\epsilon}(3\boldsymbol{H}^2 + 2{\rm Tr}(\boldsymbol{H}^2)).$$

With this value for $\boldsymbol{\delta Z}_H$ we can immediately find the one-loop $\boldsymbol{\beta}$ function for $\boldsymbol{H}$,

$$\overline{\mu}\frac{{\rm d}\boldsymbol{H}}{{\rm d}\overline{\mu}} = \boldsymbol{\beta} = -\frac{\epsilon}{2}\boldsymbol{H} + \frac{1}{16\pi^2}\left(3\boldsymbol{H}^3 + 2{\rm Tr}(\boldsymbol{H}^2)\boldsymbol{H}\right) + \mathcal{O}(\boldsymbol{H}^5). \quad (30)$$

$\boldsymbol{\beta}$, which does not depend on λ at one loop, commutes with $\boldsymbol{H}$. This need not be the case in more complicated theories where $\boldsymbol{\beta}$ can depend on other dimensionless matrices.

With $\boldsymbol{\delta M}$ and $\boldsymbol{\beta}$, from (27) and (30), we obtain the evolution equation for $\boldsymbol{M}$,

$$\overline{\mu}\frac{{\rm d}\boldsymbol{M}}{{\rm d}\overline{\mu}} = -\boldsymbol{\gamma}_m = \frac{1}{16\pi^2}\left(2\boldsymbol{H}\boldsymbol{M}\boldsymbol{H} + \frac{1}{2}\{\boldsymbol{M}, \boldsymbol{H}^2\}\right) + \mathcal{O}(\boldsymbol{H}^4). \quad (31)$$

The matrix $\boldsymbol{\gamma}_m$ defined by this equation is unconventional in that it has mass dimension one. At one-loop level we can take $\boldsymbol{M}$ on the r.h.s. to be diagonal (in fact, we can set $\boldsymbol{M} = \boldsymbol{M}_{\rm ph}$) and set $\boldsymbol{H}$ to its tree-level value, neglecting its $\overline{\mu}$ dependence. The solution to (31) up to terms of $\mathcal{O}(\boldsymbol{H}^4)$ is then

$$\boldsymbol{M}(\overline{\mu}) = \boldsymbol{M}(\overline{\mu}_0) - \boldsymbol{\gamma}_m \log\left(\frac{\overline{\mu}}{\overline{\mu}_0}\right). \quad (32)$$

We see that we can diagonalize the one-loop mass matrix at a given scale $\overline{\mu}_0$. If $[\boldsymbol{M}, \boldsymbol{H}] = 0$, then $\boldsymbol{M}$ also commutes with $\boldsymbol{\gamma}_m$ and we can diagonalize $\boldsymbol{M}$ at all scales. This is the case, of course, if $\boldsymbol{M} \propto \boldsymbol{1}$ or $\boldsymbol{H} \propto \boldsymbol{1}$. It is easy to check that the expression (29) for $\boldsymbol{M}$ is a solution to (31).

We consider, finally, the anomalous dimensions of fields. From the expressions for δZ_ϕ and $\boldsymbol{\delta Z}$ we find

$$\overline{\mu}\frac{{\rm d}\phi}{{\rm d}\overline{\mu}} = -\frac{1}{8\pi^2}{\rm Tr}\left(\boldsymbol{H}^2\right)\phi, \quad \overline{\mu}\frac{{\rm d}\boldsymbol{\psi}}{{\rm d}\overline{\mu}} = -\frac{1}{32\pi^2}\boldsymbol{H}^2\boldsymbol{\psi}. \quad (33)$$

Once again, the dependency of $\boldsymbol{H}$ on $\overline{\mu}$ can be neglected in the r.h.s. of these equations. The evolution of $\boldsymbol{\psi}$ with $\overline{\mu}$ is given by an hermitian matrix, which is consistent with $\boldsymbol{U} = \boldsymbol{1}$.

3 Normal mixing matrix

The treatment given in Sect. 2 for the case of a hermitian mixing matrix can be readily generalized to normal mixing matrices. Such a generalization is natural, since it comprises all mixing matrices that can be diagonalized by a unitary transformation of fields. Consider, for instance, a Yukawa interaction of the form

$$\mathcal{L} = -\phi_0^\dagger(\Box + m_{\phi_0}{}^2)\phi_0 + \overline{\boldsymbol{\psi}}_0({\rm i}\partial\!\!\!/ - \boldsymbol{M}_0)\boldsymbol{\psi}_0 + \overline{\boldsymbol{\psi}}_0\boldsymbol{N}_0\boldsymbol{\psi}_0\phi_0 + \overline{\boldsymbol{\psi}}_0\boldsymbol{N}_0^\dagger\boldsymbol{\psi}_0\phi_0^\dagger - V\left(\phi, \phi^\dagger\right), \quad (34)$$

where V contains cubic and quartic terms and $[\boldsymbol{N}, \boldsymbol{N}^\dagger] = 0$. This interaction induces ϕ–$\phi^\dagger$ mixing, which must be taken into account when renormalizing the theory. Besides that extra complication, the renormalization can be carried out along the same lines as in Sect. 2, where now

$$\boldsymbol{N}_0 = \boldsymbol{U}_N^\dagger\boldsymbol{Z}_N\boldsymbol{N}\boldsymbol{U}_N, \quad (35)$$

with $\boldsymbol{U}_N$ unitary and $\boldsymbol{Z}_N$ normal such that $[\boldsymbol{Z}_N, \boldsymbol{N}] = 0$. Unlike the previous case, though, we cannot parameterize $\boldsymbol{N}_0$ in the form $\boldsymbol{B}^\dagger\boldsymbol{N}\boldsymbol{B}$, since congruence transformations do not preserve normality.

More generally, we must ask whether the normality of $\boldsymbol{N}$ is stable under renormalization. (The hermiticity of $\boldsymbol{H}$ in Sect. 2 was obviously stable by unitarity.) That this is so can be seen by noticing that the Yukawa interaction in (34) is invariant under a $(U(1))^N$ symmetry which, in a flavor basis in which $\boldsymbol{N}$ is diagonal, is given by

$\psi_a \to \mathrm{e}^{-\mathrm{i}\alpha_a}\psi_a$, $\phi \to \phi$. This $(U(1))^N$ symmetry will generally be broken by fermion mass terms. In $\overline{\mathrm{MS}}$ the Yukawa coupling will remain normal after radiative corrections are taken into account, since the fermion wave-function renormalization matrix is diagonal in the interaction basis due to the $(U(1))^N$ symmetry (or, in this model, because it must be a polynomial in $\boldsymbol{N}$). In OS finite asymmetric counterterms to the Yukawa coupling are needed, which render the renormalized Lagrangian asymmetric. An example of this well-known phenomenon (see, e.g., Chapter 4 of [10] and references therein) is considered in the next section in connection with unitary mixing.

4 Unitary mixing matrix

The case of a unitary mixing matrix constitutes an important example of non-hermitian, normal mixing. Besides its obvious phenomenological relevance [1,5,6], this case is interesting because mixing-matrix unitarity imposes restrictive constraints on the form of counterterms, on top of those stemming from normality.

Furthermore, by considering unitary mixing we can extend our approach to interactions of the general form $\overline{\boldsymbol{\psi}}_1 \boldsymbol{V} \boldsymbol{\psi}_2$, with two different families of fermions. Independent unitary transformations of the fermion fields lead to a biunitary transformation of the mixing matrix $\boldsymbol{V}$. In general normality is not preserved by biunitary transformations, so our previous results do not apply to this type of interactions unless $\boldsymbol{V}$ is unitary.

In this case, the $(U(1))^N$ flavor symmetry of the previous section is replaced by a larger $SU(N)$ invariance, since we can choose the phases of fermion fields so that in the interaction basis the mixing matrix is just the identity. For concreteness we consider the simple Lagrangian, written in the interaction basis,

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_\gamma + \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_Y + \mathcal{L}_{\mathrm{em}} - V,\\
\mathcal{L}_\gamma &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\gamma^2 A_\mu A^\mu - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2,\\
\mathcal{L}_\phi &= -\phi^\dagger(\Box + m_\phi^2)\phi,\\
\mathcal{L}_\psi &= \sum_{j=1}^{2}\overline{\boldsymbol{\psi}}_j\left(\mathrm{i}\partial\!\!\!/ - P_+\boldsymbol{M}_j - P_-\boldsymbol{M}_j^\dagger\right)\boldsymbol{\psi}_j,\\
\mathcal{L}_Y &= g\overline{\boldsymbol{\psi}}_1 P_+ \boldsymbol{\psi}_2\phi + g\overline{\boldsymbol{\psi}}_2 P_- \boldsymbol{\psi}_1\phi^\dagger,\\
\mathcal{L}_{\mathrm{em}} &= Q_\phi^2 e^2\phi^\dagger\phi A_\mu A^\mu + \left(-j_\phi^\mu + j_1^\mu + j_2^\mu\right)A_\mu,\\
V &= \frac{\lambda}{6}\left(\phi^\dagger\phi\right)^2,\\
j_\phi^\mu &= Q_\phi e\mathrm{i}\left(\partial^\mu\phi^\dagger\phi - \phi^\dagger\partial^\mu\phi\right),\\
j_j^\mu &= Q_j e\overline{\boldsymbol{\psi}}_j\gamma^\mu\boldsymbol{\psi}_j. \qquad (36)
\end{aligned}$$

The model consists of two fermion families $\boldsymbol{\psi}_{1,2}$, each containing N flavors. These interact with a scalar field ϕ trough the Yukawa couplings in $\mathcal{L}_Y$, which are arbitrarily chosen to be chiral, with $P_\pm = (1 \pm \gamma_5)/2$. We choose g to be real and positive, since its phase can always be absorbed in ϕ. All of these fields are charged, with $Q_\phi = Q_1 - Q_2$, and minimally coupled to a massive photon field A^μ. $U(1)$ gauge invariance is broken explicitly by the photon mass and by the covariant gauge fixing terms. One-loop radiative corrections to the Yukawa interaction arise only from the couplings of $\boldsymbol{\psi}_j$ and ϕ to the massive photon.

Besides $U(1)$ gauge invariance, $\mathcal{L}$ has a global $SU(N)$ flavor symmetry $\boldsymbol{\psi}_j \to \mathrm{e}^{-\mathrm{i}\sum_A \alpha_A \boldsymbol{\lambda}_A}\boldsymbol{\psi}_j$, broken by fermion mass differences, with associated current

$$\begin{aligned}
j_A^\mu &= \sum_{j=1}^{2}\overline{\boldsymbol{\psi}}_j\gamma^\mu\boldsymbol{\lambda}_A\boldsymbol{\psi}_j,\\
\partial_\mu j_A^\mu &= \mathrm{i}\sum_{j=1}^{2}\overline{\boldsymbol{\psi}}_j\left(P_+[\boldsymbol{M}_j,\boldsymbol{\lambda}_A] + P_-[\boldsymbol{M}_j^\dagger,\boldsymbol{\lambda}_A]\right)\boldsymbol{\psi}_j. \qquad (37)
\end{aligned}$$

Here, $\boldsymbol{\lambda}_A$ ($A = 1,\ldots,N^2-1$) are the generators of $SU(N)$ in the fundamental representation. $\mathcal{L}$ has also broken global axial $U(1)_5$ and $SU(N)_5$ symmetries which will not be needed in the sequel.

The renormalization of this model proceeds much in the same way as in Sect. 2. We introduce renormalization constants Z_ϕ, δm_ϕ^2 for the ϕ field and its mass. For the Yukawa coupling we write $g_0 = \mu^{\epsilon/2}Z_g g$, and for the scalar self-coupling, $\lambda_0 = \mu^\epsilon(\lambda + \delta\lambda)$. By gauge invariance, only wave-function renormalization is needed for the gauge sector [8,11],

$$\begin{aligned}
&A_0^\mu = Z_\gamma^{1/2}A^\mu, \quad m_{\gamma 0}^2 = Z_\gamma m_\gamma^2, \quad \xi_0 = Z_\gamma\xi,\\
&e_0 = \mu^{\epsilon/2}Z_\gamma^{-1/2}e.
\end{aligned}$$

Henceforth, we work in a flavor basis such that at tree level mass matrices are diagonal. We can always find unitary matrices $\boldsymbol{V}_j$, $\boldsymbol{W}_j$, $(j = 1,2)$, so that the mass matrices are written as $\boldsymbol{M}_j = \boldsymbol{V}_j\boldsymbol{M}'_j\boldsymbol{W}_j^\dagger$, with $\boldsymbol{M}'_j$ diagonal, with positive entries; see Sect. 21.3 in [12]. The corresponding transformation of fermion fields is $\boldsymbol{\psi}_j = (\boldsymbol{W}_j P_+ + \boldsymbol{V}_j P_-)\boldsymbol{\psi}'_j$. Writing the Lagrangian in terms of primed quantities, and dropping the primes, at tree level we get

$$\begin{aligned}
\mathcal{L}_\psi &= \sum_{j=1}^{2}\overline{\boldsymbol{\psi}}_j\left(\mathrm{i}\partial\!\!\!/ - \boldsymbol{M}\right)\boldsymbol{\psi}_j,\\
\mathcal{L}_Y &= g\overline{\boldsymbol{\psi}}_1 P_+ \boldsymbol{V}\boldsymbol{\psi}_2\phi + g\overline{\boldsymbol{\psi}}_2 P_- \boldsymbol{V}^\dagger\boldsymbol{\psi}_1\phi^\dagger, \qquad (38)
\end{aligned}$$

with $\boldsymbol{V} = \boldsymbol{V}_1^\dagger\boldsymbol{W}_2$ the unitary mixing matrix. In this basis, the $SU(N)$ global flavor symmetry is given by

$$\begin{aligned}
\boldsymbol{\psi}_1 &\to \mathrm{e}^{-\mathrm{i}\sum_A \alpha_A\boldsymbol{\lambda}_A}\boldsymbol{\psi}_1,\\
\boldsymbol{\psi}_2 &\to \boldsymbol{V}^\dagger\mathrm{e}^{-\mathrm{i}\sum_A \alpha_A\boldsymbol{\lambda}_A}\boldsymbol{V}\boldsymbol{\psi}_2, \qquad (39a)\\
j_A^\mu &= \overline{\boldsymbol{\psi}}_1\gamma^\mu\boldsymbol{\lambda}_A\boldsymbol{\psi}_1 + \overline{\boldsymbol{\psi}}_2\gamma^\mu\boldsymbol{V}^\dagger\boldsymbol{\lambda}_A\boldsymbol{V}\boldsymbol{\psi}_2,\\
\partial_\mu j_A^\mu &= \mathrm{i}\overline{\boldsymbol{\psi}}_1[\boldsymbol{M}_1,\boldsymbol{\lambda}_A]\boldsymbol{\psi}_1 + \mathrm{i}\overline{\boldsymbol{\psi}}_2[\boldsymbol{M}_2,\boldsymbol{V}^\dagger\boldsymbol{\lambda}_A\boldsymbol{V}]\boldsymbol{\psi}_2. \qquad (39b)
\end{aligned}$$

Notice that we can parameterize this flavor symmetry, and its associated currents, in an infinite number of different ways. We could have written, for example,

$$\psi_j \to \left(P_+ \boldsymbol{V}^{1/2\dagger} \exp\left(-\mathrm{i}\sum_A \alpha_A \boldsymbol{\lambda}_A\right) \boldsymbol{V}^{1/2} + P_- \boldsymbol{V}^{1/2} \exp\left(-\mathrm{i}\sum_A \alpha_A \boldsymbol{\lambda}_A\right) \boldsymbol{V}^{1/2\dagger} \right) \boldsymbol{\psi_j}.$$

Using the same argument as above, we can apply a finite unitary renormalization of fermion fields so that $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are hermitian in each order of perturbation theory. Assuming this has been done, we can use (A.3b) to write

$$\boldsymbol{M}_{j0} = \boldsymbol{U}^\dagger_{mjL}(\boldsymbol{M}_j + \overline{\boldsymbol{\delta}}\boldsymbol{M}_j)\boldsymbol{U}_{mjR}, \tag{40}$$

so that in $\mathcal{L}_\psi$

$$\begin{aligned}(P_+\boldsymbol{M}_{j0} + P_-\boldsymbol{M}^\dagger_{j0}) &= \overline{\boldsymbol{\mathcal{U}}}_{mj}(\boldsymbol{M}_j + \overline{\boldsymbol{\delta}}\boldsymbol{M}_j)\boldsymbol{\mathcal{U}}_{mj}, \quad \text{with}\\ \boldsymbol{\mathcal{U}}_{mj} &= P_+\boldsymbol{U}_{mjR} + P_-\boldsymbol{U}_{mjL},\\ \overline{\boldsymbol{\mathcal{U}}}_{mj} &= \gamma^0\boldsymbol{\mathcal{U}}^\dagger_{mj}\gamma^0,\\ \boldsymbol{\mathcal{U}}_{mj}\boldsymbol{\mathcal{U}}^\dagger_{mj} &= \boldsymbol{1} = \boldsymbol{\mathcal{U}}^\dagger_{mj}\boldsymbol{\mathcal{U}}_{mj}.\end{aligned} \tag{41}$$

In OS these expressions are always valid since $\boldsymbol{M}_j$ are real diagonal to all orders, and therefore hermitian. In $\overline{\mathrm{MS}}$ we can use (40) at one loop, because the tree-level $\boldsymbol{M}_j$ have been chosen hermitian. The resulting one-loop renormalized mass matrices will not be hermitian. As mentioned above, we can restore hermiticity at one loop, at a given renormalization scale, by means of a finite unitary renormalization of fermion fields. This is analogous to the choice of one-loop off-diagonal entries of the mass matrix in Sect. 2.4.

The bare fermion fields are given in terms of renormalized ones by expressions similar to (4):

$$\psi_{j0} = \boldsymbol{\mathcal{U}}_j \boldsymbol{\mathcal{Z}}_j^{1/2}\psi_j, \tag{42}$$

$$\begin{aligned}\boldsymbol{\mathcal{Z}}_j^{1/2} &= (P_+\boldsymbol{Z}_{jR}^{1/2} + P_-\boldsymbol{Z}_{jL}^{1/2}),\\ \boldsymbol{\mathcal{U}}_j &= (P_+\boldsymbol{U}_{jR} + P_-\boldsymbol{U}_{jL}), \quad \boldsymbol{\mathcal{U}}_j\boldsymbol{\mathcal{U}}_j^\dagger = \boldsymbol{1}.\end{aligned} \tag{43}$$

The mixing matrix $\boldsymbol{V}$ introduced in (38) is renormalized according to (A.1),

$$\boldsymbol{V}_0 = \boldsymbol{W}^\dagger\boldsymbol{Z}_V\boldsymbol{V}\boldsymbol{W}, \quad [\boldsymbol{Z}_V, \boldsymbol{V}] = 0, \quad \boldsymbol{W} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\delta W}}. \tag{44}$$

Since $\boldsymbol{Z}_V$ is unitary, we can write $\boldsymbol{Z}_V = \exp(-\mathrm{i}\boldsymbol{\delta Z}_V)$, with $\boldsymbol{\delta Z}_V$ hermitian, $[\boldsymbol{\delta Z}_V, \boldsymbol{V}] = 0$. Other parameterizations for the mixing-matrix counterterms are discussed in the next section.

Substituting these expressions for bare quantities in $\mathcal{L}$, we obtain its expression in terms of renormalized parameters and fields. The fermion and Yukawa Lagrangians, in particular, read

$$\begin{aligned}\mathcal{L}_\psi + \mathcal{L}_Y = &\sum_j \overline{\psi}_j \overline{\boldsymbol{\mathcal{Z}}}_j^{1/2}\mathrm{i}\partial\!\!\!/\boldsymbol{\mathcal{Z}}_j^{1/2}\psi_j\\ &- \sum_j \overline{\psi}_j \overline{\boldsymbol{\mathcal{Z}}}_j^{1/2}\overline{\boldsymbol{\mathcal{U}}}_j\overline{\boldsymbol{\mathcal{U}}}_{mj}(\boldsymbol{M} + \overline{\boldsymbol{\delta}}\boldsymbol{M}_j)\\ &\times\boldsymbol{\mathcal{U}}_{mj}\boldsymbol{\mathcal{U}}_j\boldsymbol{\mathcal{Z}}_j^{1/2}\psi_j + \mu^{\epsilon/2}Z_g Z_\phi^{1/2} g\overline{\psi}_1 P_+ \boldsymbol{Z}_{1L}^{1/2}\\ &\times\boldsymbol{U}^\dagger_{1L}\boldsymbol{W}^\dagger\boldsymbol{Z}_V\boldsymbol{V}\boldsymbol{W}\boldsymbol{U}_{2R}\boldsymbol{Z}_{2R}^{1/2}\psi_2\phi + \text{H.c.}\end{aligned} \tag{45}$$

Just as in Sect. 2, out of the unitary matrices $\boldsymbol{\mathcal{U}}_j, \boldsymbol{\mathcal{U}}_{mj}, \boldsymbol{W}$ that we have introduced, only the combinations $\boldsymbol{\mathcal{U}}_{mj}\boldsymbol{\mathcal{U}}_j$, $\boldsymbol{W}\boldsymbol{U}_{1L}$ and $\boldsymbol{W}\boldsymbol{U}_{2R}$ enter $\mathcal{L}$. Clearly, we can always choose $\boldsymbol{\mathcal{U}}_j = 1$. In OS, however, it is convenient to set $\boldsymbol{\mathcal{U}}_{mj} = \boldsymbol{1}$ instead, so that bare mass matrices are diagonal in the mass basis.

We see from (45) that for the interaction term to retain its form, with a unitary mixing matrix, we need $\boldsymbol{Z}_{1L}$, $\boldsymbol{Z}_{2R} \propto \boldsymbol{1}$. This is the case in $\overline{\mathrm{MS}}$, due to $SU(N)$ flavor symmetry. In OS finite asymmetric counterterms to the Yukawa coupling are needed, so the form of the Lagrangian is not preserved.

4.1 Other parameterizations for the mixing matrix

The parameterization of counterterms given in (44) conforms to the general form for normal matrices given in (A.1). Since $\boldsymbol{V}_0$ and $\boldsymbol{V}$ are both unitary, however, we can write the relation between them in other ways. Clearly, $\boldsymbol{V}_0 = \boldsymbol{W}\boldsymbol{V}$ or $\boldsymbol{V}_0 = \boldsymbol{V}\boldsymbol{W}$ are admissible since we can reach any unitary matrix in a neighborhood of $\boldsymbol{V}$ by varying $\boldsymbol{W}$ over a neighborhood of the identity in $SU(N)$. Another usual way of writing the renormalization constants for $\boldsymbol{V}$ is $\boldsymbol{V}_0 = \boldsymbol{W}_1\boldsymbol{V}\boldsymbol{W}_2$, with $\boldsymbol{W}_{1,2}$ unitary [1,4]. This parameterization is convenient from the calculational point of view. Here we point out that it is overspecified; it must satisfy constraint relations analogous to those considered in Sect. 2.1, as we show next.

Given a map $\boldsymbol{F} : SU(N) \to SU(N)$ of the form $\boldsymbol{F}(\boldsymbol{V}) = \boldsymbol{W}_1\boldsymbol{V}\boldsymbol{W}_2$, $\boldsymbol{W}_j(\boldsymbol{V}) \in SU(N)$, we can always find $\widetilde{\boldsymbol{W}}_{1,2}$ such that

$$\begin{aligned}&\boldsymbol{F}(\boldsymbol{V}) = \widetilde{\boldsymbol{W}}_1\boldsymbol{V}\widetilde{\boldsymbol{W}}_2, \quad \widetilde{\boldsymbol{W}}_j = \widetilde{\boldsymbol{W}}_j(\boldsymbol{V}) \in SU(N),\\ &[\widetilde{\boldsymbol{W}}_2\widetilde{\boldsymbol{W}}_1, \boldsymbol{V}] = 0 = [\widetilde{\boldsymbol{W}}_1\widetilde{\boldsymbol{W}}_2, \boldsymbol{F}(\boldsymbol{V})].\end{aligned} \tag{46}$$

To see this, we notice that, given $\boldsymbol{W}_{1,2}$, we can use the lemma 1 of Appendix A to write $\boldsymbol{F}(\boldsymbol{V})$ as in (44) (or (A.1)). With the same notation as in (44), setting $\widetilde{\boldsymbol{W}}_1 = \boldsymbol{W}^\dagger\boldsymbol{Z}_V$ and $\widetilde{\boldsymbol{W}}_2 = \boldsymbol{W}$ we get (46).

4.2 On-shell scheme

We consider only the case of regular mass matrices $\boldsymbol{M}_j$. The extension to degenerate $\boldsymbol{M}_j$ can be carried out as in Sect. 2.3. Renormalization conditions in this scheme break flavor symmetry, so counterterms are not symmetric, their finite parts being tuned so the field basis is such that $\boldsymbol{M}_j$ are diagonal to all orders. Setting $\boldsymbol{\mathcal{U}}_{mj} = \boldsymbol{1}$ in (41) we have $\boldsymbol{M}_{j0} = \boldsymbol{M}_j + \overline{\boldsymbol{\delta}}\boldsymbol{M}_j$, with $\overline{\boldsymbol{\delta}}\boldsymbol{M}_j$ diagonal. To one-loop order we then have

$$\mathcal{L}_\psi = \sum_j \overline{\psi}_j \mathrm{i}\partial\!\!\!/\psi_j + \sum_j \overline{\psi}_j \mathrm{i}\partial\!\!\!/ \left(P_+\delta\boldsymbol{Z}_{jR} + P_-\delta\boldsymbol{Z}_{jL}\right)\psi_j$$

$$
\begin{aligned}
&- \sum_j \overline{\psi}_j \boldsymbol{M}_j \psi_j - \sum_j \overline{\psi}_j \left(P_+ \boldsymbol{\Delta M}_j + P_- \boldsymbol{\Delta M}_j^\dagger \right) \psi_j, \\
\mathcal{L}_Y &= \mu^{\epsilon/2} g \overline{\psi}_1 P_+ \left(\boldsymbol{V} + \boldsymbol{\Delta V} + \boldsymbol{V} \Delta g + \boldsymbol{\Delta \Gamma} \right) \psi_2 \phi + \text{H.c.} \\
\mathcal{L}_{\text{em}} &= Q_\phi^2 \mu^\epsilon e^2 \left(1 + \text{Re}(\delta Z_\phi) \right) \phi^\dagger \phi A_\mu A^\mu \\
&\quad - \mu^{\epsilon/2} j_\phi^\mu \left(1 + \text{Re}(\delta Z_\phi) \right) A_\mu + \sum_j \mu^{\epsilon/2} j_j^\mu A_\mu \\
&\quad + \sum_j Q_j \mu^{\epsilon/2} e \overline{\psi}_j \gamma^\mu \left(P_- \boldsymbol{\delta Z}_{jL} + P_+ \boldsymbol{\delta Z}_{jR} \right) \psi_j A_\mu, \quad (47)
\end{aligned}
$$

where the counterterms are defined as

$$
\begin{aligned}
\boldsymbol{\Delta M}_j &= \overline{\boldsymbol{\delta}} \boldsymbol{M}_j + \text{i} \boldsymbol{\delta U}_{jL} \boldsymbol{M}_j - \text{i} \boldsymbol{M}_j \boldsymbol{\delta U}_{jR} + \frac{1}{2} \boldsymbol{\delta Z}_{jL} \boldsymbol{M}_j \\
&\quad + \frac{1}{2} \boldsymbol{M}_j \boldsymbol{\delta Z}_{jR}, \quad (48) \\
\boldsymbol{\Delta V} &= \text{i}[\boldsymbol{\delta W}, \boldsymbol{V}] - \text{i} \boldsymbol{\delta Z}_V \boldsymbol{V} + \text{i} \boldsymbol{\delta U}_{1L} \boldsymbol{V} - \text{i} \boldsymbol{V} \boldsymbol{\delta U}_{2R}, \\
\Delta g &= \delta Z_g + \frac{1}{2} \delta Z_\phi, \quad \boldsymbol{\Delta \Gamma} = \frac{1}{2} \boldsymbol{\delta Z}_{1L} \boldsymbol{V} + \frac{1}{2} \boldsymbol{V} \boldsymbol{\delta Z}_{2R}.
\end{aligned}
$$

The flavor-asymmetric counterterms must be finite, divergent terms being flavor symmetric. In particular, the divergent parts of wave-function renormalization $(\boldsymbol{\delta Z}_{jL,R})_{\text{div}} \propto \boldsymbol{1}$. The phase of δZ_ϕ appears only in Δg, so it can be adjusted to keep Δg real. We defined $\boldsymbol{\Delta V}$ so that it is perturbatively unitary. In a flavor-symmetric, mass-independent scheme such as $\overline{\text{MS}}$, fermion wave-function renormalization constants are flavor scalars that can be absorbed in Δg, no other counterterms to the Yukawa coupling being needed. In particular, there are no flavor-breaking dimension 4 operators in $\mathcal{L}$. In OS, however, we also need $\boldsymbol{\Delta \Gamma}$, whose finite part can be viewed as either breaking the unitarity of $\boldsymbol{V}$ or the scalar nature of g.

For the computation of Feynman graphs involving fermion loops with γ_5 vertices we use 't Hooft and Veltman's prescription, with γ_5 anticommuting with γ^μ for $\mu = 0, \ldots, 3$ and commuting otherwise. We henceforth set $Q_\phi = 0$ for the sake of simplicity. For the scalar field self-energy, with OS renormalization conditions, we find

$$
\begin{aligned}
\Pi_\phi(p^2) &= \Omega_\phi(p^2) - \Omega_\phi(m_\phi^2) - (p^2 - m_\phi^2) \Omega'_\phi(m_\phi^2), \\
\text{Re}(\delta Z_\phi) &= -\frac{g^2 N}{16\pi^2} \left(\frac{2}{\epsilon} - \frac{1}{3} \right) + \Omega'_\phi(m_\phi^2), \\
\delta m_\phi^2 &= -\frac{g^2}{16\pi^2} \left(\frac{4}{\epsilon} \left(\sum_{ja} m_{ja}^2 - \frac{N}{2} m_\phi^2 \right) \right. \\
&\quad \left. + \frac{N}{3} m_\phi^2 - \sum_{ja} m_{ja}^2 (1 + a_0(m_{ja}^2)) \right) + \Omega_\phi(m_\phi^2), \\
\Omega_\phi(p^2) &= \frac{g^2}{16\pi^2} \sum_{a,b} \boldsymbol{V}_{ab} \boldsymbol{V}_{ba}^\dagger (p^2 - m_{1a}^2 - m_{2b}^2) \\
&\quad \times b_0(p^2, m_{1a}^2, m_{2b}^2). \quad (49)
\end{aligned}
$$

The fermion two-point function for each family can be expressed in terms of form factors as

$$
\begin{aligned}
\boldsymbol{\Gamma}_j(p^2) &= \not{p} - \boldsymbol{M}_j - \boldsymbol{\Pi}_j(p^2), \\
\boldsymbol{\Pi}_j(p) &= (\not{p} \boldsymbol{\Sigma}_j^L(p^2) + \boldsymbol{\Delta M}_j^\dagger) P_- \\
&\quad + (\not{p} \boldsymbol{\Sigma}_j^R(p^2) + \boldsymbol{\Delta M}_j) P_+ + \boldsymbol{\Sigma}_j^S(p^2). \quad (50)
\end{aligned}
$$

The form factors for the first family are given by

$$
\begin{aligned}
\Sigma_{1ab}^L(p^2) &= -\delta Z_{1ab}^L - \frac{g^2}{16\pi^2 \epsilon} \delta_{ab} - Q_1^2 e^2 \frac{1+\xi}{8\pi^2 \epsilon} \delta_{ab} \\
&\quad + \Omega_{1ab}^Y(p^2) + Q_1^2 \Omega_{\text{em}}^V(p^2, m_\gamma^2, m_{1a}^2) \delta_{ab}, \\
\Sigma_{1ab}^R(p^2) &= -\delta Z_{1ab}^R - Q_1^2 e^2 \frac{1+\xi}{8\pi^2 \epsilon} \delta_{ab} \\
&\quad + Q_1^2 \Omega_{\text{em}}^V(p^2, m_\gamma^2, m_{1a}^2) \delta_{ab}, \\
\Sigma_{1ab}^S(p^2) &= m_{1a} Q_1^2 e^2 \frac{3+\xi}{8\pi^2 \epsilon} \delta_{ab} \\
&\quad + m_{1a} Q_1^2 \Omega_{\text{em}}^S(p^2, m_\gamma^2, m_{1a}^2) \delta_{ab}. \quad (51)
\end{aligned}
$$

It is not difficult to check that the dependence on ξ in the scalar and fermion propagators satisfies $U(1)_{\text{e.m.}}$ Ward identities to one-loop level; see Sect. 18.7 in [11]. Notice the flavor dependence of e.m. corrections in (51). This flavor structure is also apparent in the counterterms to the e.m. current in (47). In (51) we have introduced the quantities

$$
\Omega_{1ab}^Y(p^2) = \frac{g^2}{16\pi^2} \sum_c V_{ac} V_{cb}^\dagger b_-(p^2, m_\phi^2, m_{2c}^2), \quad (52a)
$$

$$
\begin{aligned}
\Omega_{\text{em}}^V(p^2, m_\gamma^2, m_c^2) &= \frac{e^2}{16\pi^2} \left(1 + 2 b_-(p^2, m_\gamma^2, m_c^2) \right. \\
&\quad - b_0(p^2, m_\gamma^2, m_c^2) + \xi b_0(p^2, \xi m_\gamma^2, m_c^2) \\
&\quad + \frac{p^2 - m_c^2}{m_\gamma^2} \big(b_1(p^2, m_\gamma^2, m_c^2) \\
&\quad \left. - b_1(p^2, \xi m_\gamma^2, m_c^2) \big) \right), \quad (52b) \\
\Omega_{\text{em}}^S(p^2, m_\gamma^2, m_c^2) &= -\frac{e^2}{16\pi^2} \left(2 + 3 b_0(p^2, m_\gamma^2, m_c^2) \right. \\
&\quad \left. + \xi b_0(p^2, \xi m_\gamma^2, m_c^2) \right). \quad (52c)
\end{aligned}
$$

The corresponding expressions for the second family are obtained by changing the family index 1→2 and $L \leftrightarrow R$, with

$$
\Omega_{2ab}^Y(p^2) = \frac{g^2}{16\pi^2} \sum_c V_{ac}^\dagger V_{cb} b_-(p^2, m_\phi^2, m_{1c}^2).
$$

The renormalization conditions are

$$
\begin{aligned}
&\Sigma_{jaa}^L(m_{ja}^2) + \Sigma_{jaa}^R(m_{ja}^2) + 2 m_{ja} \frac{\partial}{\partial p^2} \\
&\quad \times \left[m_{ja} \Sigma_{jaa}^L(p^2) + m_{ja} \Sigma_{jaa}^R(p^2) + 2 \Sigma_{jaa}^S(p^2) \right]_{p^2 = m_{ja}^2} \\
&\quad = 0, \quad (53a)
\end{aligned}
$$

for diagonal functions and

$$
m_{jb} \Sigma_{jab}^{L,R}(m_{1b}^2) + \Delta M_{jab}^{R,L} + \Sigma_{jab}^S(m_{jb}^2) = 0,
$$

$$m_{ja}\Sigma^{L,R}_{jab}(m^2_{1a}) + \Delta M^{L,R}_{jab} + \Sigma^{S}_{jab}(m^2_{ja}) = 0, \quad (53b)$$

for both diagonal and off-diagonal ones [4]. Here we have used the notation $\Delta M^R_{jab} \equiv \Delta M_{jab}$ and $\Delta M^L_{jab} \equiv \Delta M^\dagger_{jab}$ for brevity. The renormalization constants are then, in the flavor-diagonal case,

$$\begin{aligned}\delta Z^R_{1aa} = &-\frac{Q_1^2 e^2 \xi}{8\pi^2\epsilon} + Q_1^2 \Omega^V_{\rm em}(m^2_{1a}, m^2_\gamma, m^2_{1a}) \\ &+ m^2_{1a}\Omega^{Y'}_{1aa}(m^2_{1a}) \\ &+ 2m^2_{1a}Q_1^2\Omega^{V'}_{\rm em}(m^2_{1a}, m^2_\gamma, m^2_{1a}) \\ &+ 2m^2_{1a}Q_1^2\Omega^{S'}_{\rm em}(m^2_{1a}, m^2_\gamma, m^2_{1a}), \quad (54a)\end{aligned}$$

$$\delta Z^L_{1aa} = -\frac{g^2}{16\pi^2\epsilon} + \Omega^Y_{1aa}(m^2_{1a}) + \delta Z^R_{1aa}, \quad (54b)$$

$$\begin{aligned}\Delta M_{1aa} = &-m_{1a}Q_1^2\Omega^S_{\rm em}(m^2_{1a}, m^2_\gamma, m^2_{1a}) - m_{1a}Q_1^2 \\ &\times\Omega^V_{\rm em}(m^2_{1a}, m^2_\gamma, m^2_{1a}) + m_{1a}\delta Z^R_{1aa}. \quad (54c)\end{aligned}$$

In the off-diagonal case e.m. contributions vanish and we obtain the simpler expressions

$$\delta Z^R_{1ab} = \frac{m_{1a}m_{1b}}{m^2_{1a} - m^2_{1b}}\left(\Omega^Y_{1ab}(m^2_{1a}) - \Omega^Y_{1ab}(m^2_{1b})\right), \quad (54d)$$

$$\delta Z^L_{1ab} = \frac{m^2_{1a}\Omega^Y_{1ab}(m^2_{1a}) - m^2_{1b}\Omega^Y_{1ab}(m^2_{1b})}{m^2_{1a} - m^2_{1b}}, \quad (54e)$$

$$\Delta M_{1ab} = m_{1a}\delta Z^R_{1ab}. \quad (54f)$$

We also obtain $\Delta M^L_{1ab} = m_{1b}\delta Z^R_{1ab}$, which is consistent with the above definitions. Notice that off-diagonal counterterms are finite, as required by flavor symmetry. Substituting these results back into (51) we obtain the renormalized fermion self-energy for the first family.

In order to compute $\boldsymbol{\delta U}^{L,R}_1$ we proceed as in Sect. 2. It is convenient to split $\boldsymbol{\Delta M}$ into its hermitian and antihermitian parts and, using (48), write an equation analogous to (16) for each part. Proceeding in that way we get

$$\overline{\delta} M_{1ab} = \delta_{ab}\frac{m_{1a}}{2}\left(\delta Z^R_{1aa} - \delta Z^L_{1aa}\right), \quad (55a)$$

$$\delta U^R_{1ab} = \frac{\mathrm{i}}{2}\frac{m^2_{1a} + m^2_{1b}}{m^2_{1a} - m^2_{1b}}\delta Z^R_{1ab} - \mathrm{i}\frac{m_{1a}m_{1b}}{m^2_{1a} - m^2_{1b}}\delta Z^L_{1ab}, \quad (55b)$$

$$\delta U^L_{1ab} = \mathrm{i}\frac{m_{1a}m_{1b}}{m^2_{1a} - m^2_{1b}}\delta Z^R_{1ab} - \frac{\mathrm{i}}{2}\frac{m^2_{1a} + m^2_{1b}}{m^2_{1a} - m^2_{1b}}\delta Z^L_{1ab}, \quad (55c)$$

where in the last two lines $a \neq b$. Diagonal elements of $\boldsymbol{\delta U}^{L,R}_1$ are not determined by renormalization conditions, and are set to vanish. Renormalization constants for the second family are given by similar expressions, after changing the family index and exchanging L and R labels.

We denote the three-point 1-PI function corresponding to the Yukawa vertex as $\boldsymbol{\Gamma}_3(p_1, p_2)$. External momenta are assumed to be on their mass shell, $p_1^2 = m^2_{1a}$, $p_2^2 = m^2_{2b}$, and $p^2_\phi = (p_1 + p_2)^2 = m^2_\phi$. $\boldsymbol{\Gamma}_3$ can be decomposed in form factors as $\boldsymbol{\Gamma}_3 = \boldsymbol{\Gamma}^+ P_+ + \boldsymbol{\Gamma}^- P_- + \boldsymbol{\Gamma}^V_{3\mu}\gamma^\mu + \boldsymbol{\Gamma}^A_{3\mu}\gamma^\mu\gamma_5 + \boldsymbol{\Gamma}^T_{3\mu\nu}\sigma^{\mu\nu}$. Only $\boldsymbol{\Gamma}^+_3$ receives divergent contributions at one loop:

$$\begin{aligned}\Gamma^+_{3,ab} = &\, g(\boldsymbol{V} + \boldsymbol{V}\Delta g + \boldsymbol{\Delta V} + \boldsymbol{\Delta\Gamma})_{ab} \\ &+ ge^2Q^2V_{ab}\mathcal{I}(p_1, p_2; m_{1a}, m_{2b}), \quad (56)\end{aligned}$$

where the loop integral $\mathcal{I}$ is defined in Appendix D. We separate in $\boldsymbol{\Delta\Gamma}$ the contributions coming from flavor-diagonal fermion wave-function renormalization constants, which are divergent and gauge dependent, from the finite, ξ-independent off-diagonal ones. Thus,

$$\begin{aligned}\Gamma^+_{3,ab} = &\, g(\boldsymbol{V} + \boldsymbol{V}\Delta g + \boldsymbol{\Delta V})_{ab} + g\left(\widehat{\boldsymbol{\Delta\Gamma}}\right)_{ab} \\ &+ gV_{ab}\mathcal{G}_{ab}, \quad (57a)\end{aligned}$$

$$\left(\widehat{\boldsymbol{\Delta\Gamma}}\right)_{ab} = \frac{1}{2}\sum_{c\neq a}\delta Z^L_{1ac}V_{cb} + \frac{1}{2}\sum_{c\neq b}V_{ac}\delta Z^R_{2cb}, \quad (57b)$$

$$\begin{aligned}\mathcal{G}_{ab} = &\frac{1}{2}\delta Z^L_{1aa} + \frac{1}{2}\delta Z^R_{2bb} \\ &+ e^2Q^2\mathcal{I}(p_1, p_2; m_{1a}, m_{2b}). \quad (57c)\end{aligned}$$

An explicit expression for $\mathcal{G}_{ab}$ is as follows:

$$\begin{aligned}\mathcal{G}_{ab} = &-\frac{g^2}{16\pi^2\epsilon} + \frac{3Q^2e^2}{8\pi^2\epsilon} \\ &+ \frac{1}{2}\left(\Omega^Y_{1aa}(m^2_{1a}) + m^2_{1a}\Omega^{Y'}_{1aa}(m^2_{1a}) + (m^2_{1a}\to m^2_{2b})\right) \\ &+ \frac{Q^2e^2}{16\pi^2}\Big(-\frac{1}{2}\Lambda(m^2_{1a}, m^2_\gamma) - \frac{1}{2}\Lambda(m^2_{2b}, m^2_\gamma) \\ &+ \Lambda_\xi(m^2_{1a}, m^2_\gamma) + \Lambda_\xi(m^2_{2b}, m^2_\gamma)\Big) \\ &+ 3Q^2e^2p_1\cdot p_2 C_0(p_\phi, p_1; m^2_\gamma, m^2_{1a}, m^2_{2b}) \\ &+ Q^2e^2p_1\cdot p_2\xi C_0(p_\phi, p_1; \xi m^2_\gamma, m^2_{1a}, m^2_{2b}), \quad (58)\end{aligned}$$

where

$$\begin{aligned}&\Lambda(m^2_{1a}, m^2_\gamma) \\ &\quad = 1 + b_1(m^2_{1a}, m^2_\gamma, m^2_{1a}) + 2b_0(m^2_{1a}, m^2_\gamma, m^2_{1a}) \\ &\qquad + 4m^2_{1a}b_0'(m^2_{1a}, m^2_\gamma, m^2_{1a}) \\ &\qquad + 4m^2_{1a}b_1'(m^2_{1a}, m^2_\gamma, m^2_{1a}), \\ &\Lambda_\xi(m^2_{1a}, m^2_\gamma) \\ &\quad = \frac{m^2_{1a}}{m^2_\gamma}\Big(b_1(m^2_{1a}, m^2_\gamma, m^2_{1a}) \\ &\qquad - b_1(m^2_{1a}, \xi m^2_\gamma, m^2_{1a})\Big), \quad (59)\end{aligned}$$

and $p_1\cdot p_2 = 1/2(m^2_\phi - m^2_{1a} - m^2_{2b})$. Some comments regarding $\mathcal{G}_{ab}$ are in order. The ultraviolet divergent terms in (58) are ξ-independent, as expected since they must be cancelled by Δg. All the remaining gauge dependence in $\mathcal{G}_{ab}$ is contained in Λ_ξ and C_0. As is well known, and as is clear from its definition in Appendix D, the triangle integral C_0 with on-shell external momenta diverges logarithmically as $m^2_\gamma \to 0$. So does Λ_ξ, since we can write

$$\Lambda_\xi(m^2, m^2_\gamma) = \frac{1}{2}\left(b_0(m^2, m^2_\gamma, m^2) - a_0(m^2_\gamma)\right)$$

$$-\frac{\xi}{2}\left(b_0(m^2,\xi m_\gamma^2,m^2)-a_0(\xi m_\gamma^2)\right),$$

with

$$b_0(m^2,m_\gamma^2,m^2)-a_0(m_\gamma^2)=\int_0^1 \mathrm{d}x\log\left(1-x+\frac{m^2}{m_\gamma^2}x^2\right)+1.$$

These infrared divergences must be cancelled in the computation of transition rates by the contribution from real-photon emission diagrams. We conclude that the infrared divergent terms must appear explicitly in the amplitude, and should not be absorbed in the finite part of counterterms. In fact, the requirements of infrared and ultraviolet finiteness, that g and g_0 should be flavor independent and $\boldsymbol{V}$ and $\boldsymbol{V_0}$ unitary, and of gauge invariance, lead us to set

$$\Delta g=-(\mathcal{G}_{ab})_{\mathrm{div}}=\frac{g^2}{16\pi^2\epsilon}-\frac{3Q^2e^2}{8\pi^2\epsilon},\quad\text{and}$$
$$\boldsymbol{\Delta V}=0. \tag{60}$$

These equations determine the renormalization constants δZ_g, $\boldsymbol{\delta W}$ and $\boldsymbol{\delta Z}_V$ through (48) and the above results for wave-function renormalization constants. Our choice is not unique, though, since the above requirements still allow the finite part of δZ_g to be redefined by adding gauge- and flavor-independent arbitrary constants. We could, for instance, define g through the total ϕ decay width, or some other inclusive process. We will not pursue such a phenomenological analysis further here.

4.3 $\overline{\mathrm{MS}}$ scheme

Renormalization in $\overline{\mathrm{MS}}$ scheme turns out, as expected, to be much simpler than in OS. The renormalized Lagrangian at one-loop level takes essentially the same form as in (47) and (48), the main differences being that now $\boldsymbol{\delta U}_j^{L,R}=\mathbf{0}$, $\boldsymbol{\delta Z}_j^{L,R}=\delta Z_j^{L,R}\mathbf{1}$, and $\boldsymbol{\delta U}_{mj}^{L,R}\neq\mathbf{0}$. The mass matrices $\boldsymbol{M}_j$ are diagonal at tree level but not necessarily diagonal, or even hermitian, at one loop. We are then led to separate diagonal elements (which are real, given our choice of tree-level flavor basis) from off-diagonal ones (which are $\mathcal{O}(g^2)$ or $\mathcal{O}(e^2)$), as discussed in Sect. 2.4.

We define $\Delta g=\delta Z_g+1/2\delta Z_\phi+1/2\delta Z_1^L+1/2\delta Z_2^R$, and set $\boldsymbol{\Delta\Gamma}=\mathbf{0}$, since obviously no asymmetric counterterms are needed in this scheme. Furthermore, $\boldsymbol{\delta W}=0$ for the same reasons as in Sect. 2.4. Renormalization constants can then be read off the corresponding expressions in Sect. 4.2. In particular,

$$\delta Z_1^L=-\frac{Q^2e^2\xi}{8\pi^2\epsilon}-\frac{g^2}{16\pi^2\epsilon}=\delta Z_2^R,$$
$$\delta Z_g=\frac{g^2(N+1)}{16\pi^2\epsilon}-\frac{3Q^2e^2}{8\pi^2\epsilon}. \tag{61}$$

Divergent contributions from the triangle diagram correcting the Yukawa vertex, which are flavor symmetric, have been absorbed in δZ_g. There are no further infinities to cancel, so that $\boldsymbol{\delta Z}_V=\mathbf{0}$, i.e., $\boldsymbol{V}$ is not renormalized.

We can easily derive renormalization group equations for the renormalized parameters and Green's functions. However, $\boldsymbol{V}$ obviously does not run at one loop in this model.

5 Concluding remarks

Unitary transformations play a distinguished role in quantum theories, as is well known. In QFT, unitary transformations in some internal "flavor" space preserve the normalization of kinetic terms and currents, in particular electromagnetic ones. It is then natural to consider the class of mixing matrices that can be diagonalized by a unitary transformation, namely, normal mixing matrices, and their renormalization properties. This is the subject of the foregoing.

We point out in Sects. 2 and 4 that the usual parameterizations for mixing-matrix counterterms [1,2,5,6] are overspecified. This fact, of course, does not make them less useful in any way, but it had not been pointed out in the previous literature. We explicitly exhibit the constraints those parameterizations must satisfy, and their solution. Furthermore, the minimal parameterization given in Appendix A can be applied also to the case of normal mixing matrices, which we do in Sect. 3.

In principle, the general case of non-singular, not necessarily normal mixing matrices, although not discussed in this paper, can also be studied in the framework proposed here for normal matrices. This is so because any complex matrix $\boldsymbol{A}$ can be decomposed as $\boldsymbol{A}=\boldsymbol{N}+\boldsymbol{T}$, with $\boldsymbol{N}$ normal and $\boldsymbol{T}$ nilpotent such that they can be simultaneously brought to diagonal and strictly triangular form, respectively, by a unitary transformation.

Writing field-strength renormalization matrices in polar components is useful in the OS scheme, as discussed in Sects. 2 and 4. At one-loop level this is the same as decomposing those matrices into their hermitian and anti-hermitian parts, as was done in [1]. As remarked in Sect. 2, the unitary and hermitian components play different roles in the theory. Furthermore, the contribution to higher orders in perturbation theory from powers of the one-loop $\boldsymbol{\delta U}$ is apparent in this way due to the exponential form of the unitary matrix $\boldsymbol{U}=\exp(\mathrm{i}\boldsymbol{\delta U})$.

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Appendix

A Mappings of normal matrices

A matrix $\boldsymbol{N}\in\mathbb{C}^{N\times N}$ is called normal if $[\boldsymbol{N},\boldsymbol{N}^\dagger]=0$ [13]. It can be shown [13] that a complex matrix can be

diagonalized by a unitary transformation if and only if it is normal. We denote by $\boldsymbol{n}(N)$ the set of $N \times N$ normal matrices. The Lie algebra $\boldsymbol{u}(N)$ of hermitian matrices, the group $U(N)$ of unitary matrices, and the abelian algebra of diagonal matrices are all contained in $\boldsymbol{n}(N)$.

Lemma 1. *Let $\boldsymbol{F} : \boldsymbol{n}(N) \to \boldsymbol{n}(N)$ be a mapping of normal matrices such that* $\operatorname{rank}(\boldsymbol{F}(\boldsymbol{X})) \leq \operatorname{rank}(\boldsymbol{X})$ *for every $\boldsymbol{X}$ in $\boldsymbol{n}(N)$. Then, we can write*

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{U}^\dagger \boldsymbol{Z}\boldsymbol{X}\boldsymbol{U}, \quad \textit{with} \quad [\boldsymbol{Z}, \boldsymbol{X}] = 0, \\ &\quad \boldsymbol{Z} = \boldsymbol{Z}(\boldsymbol{X}) \in \boldsymbol{n}(N), \quad \boldsymbol{U} = \boldsymbol{U}(\boldsymbol{X}) \in U(N). \end{aligned} \tag{A.1}$$

Proof. We first consider the case of $\boldsymbol{X}$ non-singular. In fact, in this case there is an open neighborhood of $\boldsymbol{X}$ in $\boldsymbol{n}(N)$ where all matrices are non-singular. We can apply the parameterization (A.1) to every matrix in that neighborhood. Using the normality of $\boldsymbol{F}(\boldsymbol{X})$ and $\boldsymbol{X}$, we can write (dropping the argument $\boldsymbol{X}$ for brevity) $\boldsymbol{F} = \boldsymbol{V}^\dagger \boldsymbol{F}' \boldsymbol{V}$ and $\boldsymbol{X} = \boldsymbol{W}^\dagger \boldsymbol{X}' \boldsymbol{W}$, with $\boldsymbol{V}$, $\boldsymbol{W}$ unitary and $\boldsymbol{F}'$, $\boldsymbol{X}'$ diagonal. Define $\boldsymbol{Z}'$ diagonal by $\boldsymbol{F}' = \boldsymbol{Z}'\boldsymbol{X}'$. Then $\boldsymbol{U} = \boldsymbol{W}^\dagger \boldsymbol{V} \in U(N)$ and $\boldsymbol{Z} = \boldsymbol{W}^\dagger \boldsymbol{Z}' \boldsymbol{W} \in \boldsymbol{n}(N)$ satisfy (A.1).

Assume now $\boldsymbol{X}$ has a null eigenvalue with multiplicity r. We can choose $\boldsymbol{W}$ so that $\boldsymbol{X}' = \operatorname{diag}(0, \dots, 0, x_{r+1}, \dots, x_n)$ with $x_j \neq 0$. Similarly, we can choose $\boldsymbol{V}$ so that $\boldsymbol{F}' = \operatorname{diag}(0, \dots, 0, f_{q+1}, \dots, f_n)$, with $f_j \neq 0$ and $q \geq r$. We then define $\boldsymbol{Z}' = \operatorname{diag}(1, \dots, 1, f_{r+1}/x_{r+1}, \dots, f_n/x_n)$ and proceed as above.

A transformation $\boldsymbol{F}$ satisfying the rank hypothesis of the lemma may be called "multiplicative," in the sense that $\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{0}$. The same argument as in the multiplicative case, with obvious changes, proves the following. Let $\boldsymbol{F} : \boldsymbol{n}(N) \to \boldsymbol{n}(N)$ be a mapping of normal matrices. Then, for any $\boldsymbol{X} \in \boldsymbol{n}(N)$ we can write

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{U}^\dagger (\boldsymbol{X} + \boldsymbol{N}) \boldsymbol{U}, \quad \text{with} \quad [\boldsymbol{N}, \boldsymbol{X}] = 0, \\ &\quad \boldsymbol{N} = \boldsymbol{N}(\boldsymbol{X}) \in \boldsymbol{n}(N), \quad \boldsymbol{U} = \boldsymbol{U}(\boldsymbol{X}) \in U(N). \end{aligned} \tag{A.2}$$

We remark that both (A.1) and (A.2) hold for mappings of hermitian matrices, $\boldsymbol{F} : \boldsymbol{u}(N) \to \boldsymbol{u}(N)$, with $\boldsymbol{Z}$ and $\boldsymbol{N}$ hermitian, and (A.1) also holds for mappings of unitary matrices, $\boldsymbol{F} : U(N) \to U(N)$, with $\boldsymbol{Z}$ unitary.

Consider, finally, a general matrix function of a hermitian matrix $\boldsymbol{F} : \boldsymbol{u}(N) \to \mathbb{C}^{N \times N}$. Using the polar decomposition $\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{R}(\boldsymbol{X}) \boldsymbol{U}_F(\boldsymbol{X})$ with $\boldsymbol{U}_F$ unitary and $\boldsymbol{R}$ hermitian, and applying (A.1) and (A.2) to $\boldsymbol{R}$, we obtain the parameterizations,

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{U}\boldsymbol{Z}\boldsymbol{X}\boldsymbol{V}, \quad \boldsymbol{U} = \boldsymbol{U}(\boldsymbol{X}) \quad \text{and} \\ &\quad \boldsymbol{V} = \boldsymbol{V}(\boldsymbol{X}) \in U(N), \quad \boldsymbol{Z} = \boldsymbol{Z}(\boldsymbol{X}) \in \boldsymbol{u}(N), \\ &[\boldsymbol{Z}, \boldsymbol{X}] = 0, \end{aligned} \tag{A.3a}$$

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{U}(\boldsymbol{X} + \boldsymbol{\delta X})\boldsymbol{V}, \quad \boldsymbol{\delta X} = \boldsymbol{\delta X}(\boldsymbol{X}) \in \boldsymbol{u}(N), \\ &[\boldsymbol{\delta X}, \boldsymbol{X}] = 0. \end{aligned} \tag{A.3b}$$

Substituting (A.3b) into the hermitian matrices $\boldsymbol{F}\boldsymbol{F}^\dagger$ and $\boldsymbol{F}^\dagger \boldsymbol{F}$, we recover (A.1) and (A.2).

B Perturbative factorization of $SU(N)$ matrices

In Sect. 2 we make use of the fact that a unitary matrix $\boldsymbol{U}$ which is close to the identity can always be factorized as $\boldsymbol{U} = \exp(\mathrm{i}\boldsymbol{\delta U}') \exp(\mathrm{i}\boldsymbol{\delta}\widetilde{\boldsymbol{U}})$, where $\boldsymbol{\delta U}'$ is hermitian and diagonal and $\boldsymbol{\delta}\widetilde{\boldsymbol{U}}$ is hermitian and has zeros on the diagonal. A sketch of the proof follows (see [14] for a globally valid factorization).

We decompose the Lie algebra of hermitian matrices as $\boldsymbol{u}(N) = \boldsymbol{a} \oplus \boldsymbol{b}$, where $\boldsymbol{a}$ is the Cartan subalgebra of diagonal matrices of $\boldsymbol{u}(N)$ and $\boldsymbol{b}$ its complementary subspace. Let $\boldsymbol{U} = \exp(\mathrm{i}\epsilon \boldsymbol{H})$, with ϵ small and $\boldsymbol{H} \in \boldsymbol{u}(N)$. We want to show that we can always find $\boldsymbol{A} = \boldsymbol{A}(\epsilon) \in \boldsymbol{a}$ and $\boldsymbol{B} = \boldsymbol{B}(\epsilon) \in \boldsymbol{b}$ such that

$$\mathrm{e}^{\mathrm{i}\epsilon \boldsymbol{A}} \mathrm{e}^{\mathrm{i}\epsilon \boldsymbol{B}} = \sum_{n=0}^{N} \frac{(\mathrm{i}\epsilon)^n}{n!} \boldsymbol{H}^n + \mathcal{O}(\epsilon^{N+1}). \tag{B.1}$$

To this end, we write

$$\boldsymbol{A} = \sum_{n=0}^{N} \frac{\epsilon^n}{n!} \boldsymbol{A}_n,$$

with $\boldsymbol{A}_n \in \boldsymbol{a}$, and analogously for $\boldsymbol{B}$, and consider the equation $\exp(\mathrm{i}\epsilon \boldsymbol{H}) = \exp(\mathrm{i}\epsilon \boldsymbol{A}) \exp(\mathrm{i}\epsilon \boldsymbol{B})$ which, using the Campbell–Baker–Hausdorff formula (see [9,14] and references therein) can be written as

$$\begin{aligned} \boldsymbol{H} = (\boldsymbol{A} + \boldsymbol{B}) + \frac{\mathrm{i}\epsilon}{2}[\boldsymbol{A}, \boldsymbol{B}] + \frac{(\mathrm{i}\epsilon)^2}{12} \big([[\boldsymbol{A}, \boldsymbol{B}], \boldsymbol{B}] \\ + [[\boldsymbol{B}, \boldsymbol{A}], \boldsymbol{A}]\big) + \cdots \end{aligned} \tag{B.2}$$

Expanding the r.h.s. in powers of ϵ, we are led to a set of recursive equations,

$$\begin{aligned} &\boldsymbol{A}_0 + \boldsymbol{B}_0 = \boldsymbol{H} \\ &\boldsymbol{A}_1 + \boldsymbol{B}_1 = \frac{1}{2}[\boldsymbol{B}_0, \boldsymbol{A}_0] \\ &\boldsymbol{A}_2 + \boldsymbol{B}_2 = [\boldsymbol{B}_1, \boldsymbol{A}_0] + [\boldsymbol{B}_0, \boldsymbol{A}_1] - \frac{1}{6}[[\boldsymbol{A}_0, \boldsymbol{B}_0], \boldsymbol{B}_0] \\ &\qquad - \frac{1}{6}[[\boldsymbol{B}_0, \boldsymbol{A}_0], \boldsymbol{A}_0], \quad \text{etc.} \end{aligned} \tag{B.3}$$

which can be solved iteratively up to the desired order. (Notice that in (B.3) we have used that $\boldsymbol{a}$ is abelian to eliminate some commutators.) Thus, $\boldsymbol{A}_0 = \operatorname{diag}(H_{11}, \dots, H_{NN})$ and $\boldsymbol{B}_0 = \boldsymbol{H} - \boldsymbol{A}_0$, $\boldsymbol{A}_1 = \boldsymbol{0}$ and $\boldsymbol{B}_1 = 1/2[\boldsymbol{B}_0, \boldsymbol{A}_0]$, and so on. More generally, the equation for the coefficients of order $N+1$ obtained from (B.2) is of the form

$$\begin{aligned} 0 = &\frac{1}{(N+1)!}(\boldsymbol{A}_{N+1} + \boldsymbol{B}_{N+1}) \\ &+ \frac{1}{2} \sum_{\substack{m,n=0 \\ m+n=N}}^{N} \frac{1}{m!n!}[\boldsymbol{A}_m, \boldsymbol{B}_n] \\ &+ \frac{1}{12} \sum_{\substack{k,m,n=0 \\ k+m+n=N-1}}^{N-1} \frac{1}{k!m!n!} \big([[\boldsymbol{A}_k, \boldsymbol{B}_m], \boldsymbol{B}_n] \end{aligned}$$

$$- [[\boldsymbol{A}_k, \boldsymbol{B}_m], \boldsymbol{A}_n]) + \cdots$$

where the ellipsis refers to higher-order commutators. Terms with K-order commutators involve $\boldsymbol{A}_j$, $\boldsymbol{B}_j$ with $j = 0, \ldots, N+1-K$, which are already known. Thus, by projecting $(\boldsymbol{A}_{N+1} + \boldsymbol{B}_{N+1})$ as given by this equation over $\boldsymbol{a}$ and $\boldsymbol{b}$ we find $\boldsymbol{A}_{N+1}$ and $\boldsymbol{B}_{N+1}$.

To summarize, let D be a neighborhood of the identity in $U(N)$, determining a corresponding neighborhood $\boldsymbol{d}$ of $\boldsymbol{0}$ in $\boldsymbol{u}(N)$, where the CBH formula holds. Then, the elements of the form $\mathrm{e}^{\boldsymbol{A}}\mathrm{e}^{\boldsymbol{B}}$, where $\boldsymbol{A} \in \boldsymbol{a}$ and $\boldsymbol{B} \in \boldsymbol{b}$ are dense in D.

C Kernel and image of the adjoint map of a normal matrix

Consider a fixed $N \times N$ normal matrix $\boldsymbol{N}$. Then, any matrix $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ can be written as

$$\boldsymbol{A} = \boldsymbol{B} + [\boldsymbol{N}, \boldsymbol{D}], \quad \text{with} \quad [\boldsymbol{N}, \boldsymbol{B}] = 0. \tag{C.1}$$

Equations (16) and (21) are of this type. Given $\boldsymbol{A}$ and $\boldsymbol{N}$, (C.1) can always be solved for $\boldsymbol{B}$ and $\boldsymbol{D}$, as we now show (see also Sect. 3.3 in [9]).

Since $\boldsymbol{N}$ is normal, it can be diagonalized by a unitary matrix. Let $\lambda_1, \ldots, \lambda_r$ $(r \leq N)$ be its eigenvalues and $d_j (j = 1, \ldots, r)$ their multiplicities. The associated orthonormal basis of $\mathbb{C}^N$ of eigenvectors of $\boldsymbol{N}$ is denoted by $\{|\lambda_j, \alpha_j\rangle\}$ $(j = 1, \ldots, r, \alpha_j = 1, \ldots, d_j)$. Then we have $\boldsymbol{N} = \sum_{i\alpha} \lambda_i |\lambda_i, \alpha\rangle\langle\lambda_i, \alpha|$, and for any $\boldsymbol{A} \in \mathbb{C}^{N\times N}$,

$$\boldsymbol{A} = \sum_{i,\alpha,j,\beta} A^{i\alpha}_{j\beta} |\lambda_i, \alpha\rangle\langle\lambda_j, \beta|. \tag{C.2}$$

We define the adjoint map associated to $\boldsymbol{N}$ as $\mathrm{ad}_N : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$, $\mathrm{ad}_N(\boldsymbol{A}) = [\boldsymbol{N}, \boldsymbol{A}]$. Then the set of orthogonal projectors $\{|\lambda_i, \alpha_i\rangle\langle\lambda_j, \beta_j|\}$ is a basis of $\mathbb{C}^{N\times N}$ of eigenvectors of ad_N, since, $[\boldsymbol{N}, |\lambda_i, \alpha_i\rangle\langle\lambda_j, \beta_j|] = (\lambda_i - \lambda_j)|\lambda_i, \alpha_i\rangle\langle\lambda_j, \beta_j|$. Therefore, we can rewrite (C.2) as

$$\begin{aligned}\boldsymbol{A} = &\sum_{i,\alpha,\beta} A^{i\alpha}_{i\beta} |\lambda_i, \alpha\rangle\langle\lambda_i, \beta| \\ &+ \left[\boldsymbol{N}, \sum_{\substack{i,\alpha,j,\beta \\ i\neq j}} A^{i\alpha}_{j\beta} \frac{1}{\lambda_i - \lambda_j} |\lambda_i, \alpha\rangle\langle\lambda_j, \beta|\right],\end{aligned} \tag{C.3}$$

where the first term on the r.h.s. obviously commutes with $\boldsymbol{N}$. This is the decomposition (C.1).

Whereas (C.3) provides an explicit solution to (C.1), a slightly broader point of view can be adopted. Consider the inner product $(\boldsymbol{A}, \boldsymbol{B}) = \mathrm{Tr}(\boldsymbol{A}^\dagger \boldsymbol{B})$ in $\mathbb{C}^{N\times N}$. By the cyclic property of the trace, $(\boldsymbol{A}, \mathrm{ad}_N(\boldsymbol{B})) = (\boldsymbol{A}, [\boldsymbol{N}, \boldsymbol{B}]) = \mathrm{Tr}(\boldsymbol{A}^\dagger[\boldsymbol{N}, \boldsymbol{B}]) = \mathrm{Tr}([\boldsymbol{N}^\dagger, \boldsymbol{A}]^\dagger \boldsymbol{B}) = ([\boldsymbol{N}^\dagger, \boldsymbol{A}], \boldsymbol{B}) = (\mathrm{ad}_{N^\dagger}(\boldsymbol{A}), \boldsymbol{B})$, so $(\mathrm{ad}_N)^\dagger = \mathrm{ad}_{N^\dagger}$. With this, using the Jacobi identity and the normality of $\boldsymbol{N}$, it is immediate that $[(\mathrm{ad}_N)^\dagger, \mathrm{ad}_N] = 0$ and thus ad_N is a normal transformation of $\mathbb{C}^{N\times N}$. Therefore the spectral theorem [13] holds for ad_N. In particular, $\mathbb{C}^{N\times N} = \mathrm{Ker}(\mathrm{ad}_N) \oplus \mathrm{Im}(\mathrm{ad}_N)$. This orthogonal decomposition shows that solutions to (C.1) exist and are unique up to addition to $\boldsymbol{D}$ of a matrix commuting with $\boldsymbol{N}$.

D Loop integrals

In this appendix we give a list of loop integrals used in the foregoing. More complete calculations can be found, e.g., in [4, 15–17]. Divergent integrals are separated in a dimensional-regularization pole term and a finite remainder. With $\overline{\mu} = \mu(4\pi\mathrm{e}^{-\gamma_\mathrm{E}})^{1/2}$ we have

$$\begin{aligned}
A_0(m^2) &= \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d} \int \mathrm{d}^d\ell \frac{1}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \\
&= -\frac{m^2}{8\pi^2\epsilon} + \frac{m^2}{16\pi^2} a_0(m^2), \\
a_0(m^2) &= \log\left(\frac{m^2}{\overline{\mu}^2}\right) - 1, \\
A_1^\mu(m^2) &= \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d} \int \mathrm{d}^d\ell \frac{\ell^\mu}{\ell^2 - m^2 + \mathrm{i}\varepsilon} = 0, \\
A_2^{\mu\nu}(m^2) &= \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d} \int \mathrm{d}^d\ell \frac{\ell^\mu \ell^\nu}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \\
&= -\frac{m^4}{32\pi^2\epsilon} g^{\mu\nu} + \frac{m^4}{64\pi^2} g^{\mu\nu} a_2(m^2), \\
a_2(m^2) &= a_0(m^2) - \frac{1}{2}, \\
A_2(m^2) &= \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d} \int \mathrm{d}^d\ell \frac{\ell^2}{\ell^2 - m^2 + \mathrm{i}\varepsilon} \\
&= m^2 A_0(m^2), \\
B_0(p^\mu, m_1^2, m_2^2) &= \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d} \int \mathrm{d}^d\ell \\
&\quad \times \frac{1}{(\ell^2 - m_1^2 + \mathrm{i}\varepsilon)\left((\ell+p)^2 - m_2^2 + \mathrm{i}\varepsilon\right)} \\
&= -\frac{1}{8\pi^2\epsilon} + \frac{1}{16\pi^2} b_0(p^2, m_1^2, m_2^2), \\
b_0(p^2, m_1^2, m_2^2) &= \int_0^1 \mathrm{d}x \log\left((1-x)\frac{m_1^2}{\overline{\mu}^2} + x\frac{m_2^2}{\overline{\mu}^2}\right. \\
&\quad \left. - x(1-x)\frac{p^2}{\overline{\mu}^2} - \mathrm{i}\varepsilon\right), \\
B_1^\mu(p^\mu, m_1^2, m_2^2) &= \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d} \int \mathrm{d}^d\ell \\
&\quad \times \frac{\ell^\mu}{(\ell^2 - m_1^2 + \mathrm{i}\varepsilon)\left((\ell+p)^2 - m_2^2 + \mathrm{i}\varepsilon\right)} \\
&= \frac{p^\mu}{16\pi^2\epsilon} - \frac{p^\mu}{16\pi^2} b_1(p^2, m_1^2, m_2^2), \\
b_1(p^2, m_1^2, m_2^2) &= \int_0^1 \mathrm{d}x x \log\left((1-x)\frac{m_1^2}{\overline{\mu}^2} + x\frac{m_2^2}{\overline{\mu}^2}\right. \\
&\quad \left. - x(1-x)\frac{p^2}{\overline{\mu}^2} - \mathrm{i}\varepsilon\right),
\end{aligned}$$

which can also be written

$$B_1^\mu(p^\mu, m_1^2, m_2^2) = p^\mu B_1(p^2, m_1^2, m_2^2),$$
$$p^2 B_1(p^2, m_1^2, m_2^2) = \frac{1}{2}\left(A_0(m_1^2) - A_0(m_2^2) - (p^2 + m_1^2 - m_2^2)B_0(p^2, m_1^2, m_2^2)\right).$$

We also use the combination $b_-(p^2, m_1^2, m_2^2) = b_0(p^2, m_1^2, m_2^2) - b_1(p^2, m_1^2, m_2^2)$:

$$C_0(p_1, p_2, m_1^2, m_2^2, m_3^2) = \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d}\int \mathrm{d}^d\ell \frac{1}{(\ell^2 - m_1^2 + \mathrm{i}\varepsilon)\left((\ell + p_2)^2 - m_2^2 + \mathrm{i}\varepsilon\right)} \times \frac{1}{((\ell - p_1 + p_2)^2 - m_3^2 + \mathrm{i}\varepsilon)}.$$

C_0 is ultraviolet finite. In Sect. 2.2 we use the following triangle integrals:

$$H_1(p_1, p_2; m_1^2, m_2^2, m_3^2) = \frac{\mathrm{i}\mu^\epsilon}{(2\pi)^d}\int \mathrm{d}^d\ell \frac{1}{\ell^2 - m_1^2 + \mathrm{i}\varepsilon}\frac{\not\ell + \not p_2 + m_2}{(\ell + p_2)^2 - m_2^2 + \mathrm{i}\varepsilon} \times \frac{\not\ell - \not p_1 + \not p_2 + m_3}{(\ell - p_1 + p_2)^2 - m_3^2 + \mathrm{i}\varepsilon},$$
$$\mathrm{Tr}(H_1(p_1, p_2; m_1^2, m_2^2, m_3^2)) = \mathrm{Tr}(1)\left(-\frac{1}{8\pi^2\epsilon} + h_1(p_1, p_2; m_1^2, m_2^2, m_3^2)\right),$$
$$h_1(p_1, p_2; m_1^2, m_2^2, m_3^2) = \frac{1}{2}b_0\left((p_1 - p_2)^2; m_1^2, m_3^2\right) + \frac{1}{2}b_0\left(p_2^2; m_1^2, m_2^2\right) + \frac{1}{2}\left((m_2 + m_3)^2 - p_1^2\right)C_0\left(p_1, p_2; m_1^2, m_2^2, m_3^2\right),$$
$$h_2(p_1, p_2; m_1^2, m_2^2) = \mathrm{Tr}(1)m_2 C_0\left(p_1, p_2; m_2^2, m_1^2, m_1^2\right),$$

with $\mathrm{Tr}(1) = 4$. Finally, in Sect. 4.2 we define

$$\mathcal{I}(p_1, p_2; m_{1a}^2, m_{2b}^2, m_\gamma^2) = \frac{1}{2}\frac{\mu^\epsilon}{(2\pi)^d}\int \mathrm{d}^d\ell \Delta_{\mu\nu}(\ell) \times \frac{\mathrm{Tr}\left(P_+\gamma^\mu(\not\ell + \not p_1 + m_{1a})P_+(\not\ell - \not p_2 + m_{2b})\gamma^\nu\right)}{((\ell + p_1)^2 - m_{1a}^2 + \mathrm{i}\varepsilon)\left((\ell - p_2)^2 - m_{2b}^2 + \mathrm{i}\varepsilon\right)}$$
$$= \frac{3 + \xi}{8\pi^2\epsilon} - \frac{1}{8\pi^2} - \frac{1}{32\pi^2}\left(3b_0(p_1^2, m_\gamma^2, m_{1a}^2) + 3b_0(p_2^2, m_\gamma^2, m_{2b}^2) + \xi b_0(p_1^2, \xi m_\gamma^2, m_{1a}^2) + \xi b_0(p_2^2, \xi m_\gamma^2, m_{2b}^2)\right) + p_1 \cdot p_2\left(3C_0(p_\phi, p_1; m_\gamma^2, m_{1a}^2, m_{2b}^2) + \xi C_0(p_\phi, p_1; \xi m_\gamma^2, m_{1a}^2, m_{2b}^2)\right).$$

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